MATH 451: Introduction to Numerical Analysis II

Instructor: Prof. Wolfgang Bangerth
bangerth@colostate.edu

Homework assignment 4 – due Friday 4/6/2018

Problem 1 (Numerical solution of a ODE). Consider the following scalar ordinary differential equation (ODE):

$$x'(t) = \frac{1}{2}x(t), \qquad\qquad x(0) = 1.$$

For this particular equation, we know the exact solution: it corresponds to the exponential growth $x(t) = e^{\frac{1}{2}t}$.

Implement codes for the following methods:

- the explicit Euler method,
- the implicit Euler method,
- the (implicit) trapezoidal (Crank-Nicolson) method,
- the (implicit) BDF-2 method. For this method, you need to bootstrap in the first time step; use the Crank-Nicolson method for this.

Then compute approximations to $x(4)$ using each of four methods and with step sizes $\Delta t = 2, 1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{32}$. Compute their respective errors $e = |x_N - x(4)|$ where x_N is the approximation to $x(4)$ at the end of the last time step.

For each method, create either a table or a Δt-vs-e graph that shows how the error decreases as the mesh size is reduced. (For the graph, you will want to consider a log-log plot.) In all cases, the error should behave as $e \approx C\ \Delta t^s$ for some C that we would like to be as small as possible, and an s that we would like to be as large as possible. Determine both C and s from your data.

Discuss which method yields the most accurate answer. **(40 points)**

Problem 2 (Explicit vs implicit). Consider the prototypical "stiff" equation we have discussed in class:

$$\begin{aligned} x_1'(t) &= -\lambda_1 x_1(t), & x_1(0) &= x_{1,0}, \\ x_2'(t) &= -\lambda_2 x_2(t) + \lambda_1 x_1(t), & x_2(0) &= x_{2,0}. \end{aligned}$$

It describes the amounts of radioactive materials 1 and 2, where material 1 decays to material 2 at a rate of λ_1, and material 2 decays to something we do not track here at a rate of λ_2.

Choose a situation in which $\lambda_1 = 1000$, $\lambda_2 = 1$, and $x_{1,0} = x_{2,0} = 1$. Let's say we are interested in the amount of each material that is left at $t = 2$. Implement a code that uses the explicit Euler method and experiment with different values for the step sizes Δt. What do you observe if Δt is large? What do you observe if it is small? Discuss your observations, and support them by graphs that show your numerical approximations $\mathbf{x}_k = (x_{1,k}, x_{2,k})^T$ at time steps k. State how small your Δt has to be for the solution to not diverge? Does this match the theory we have discussed? **(20 points)**

Problem 3 (Explicit vs implicit). Repeat the previous problem but use the *implicit* Euler method. You will now have to solve a system of equations for $\mathbf{x}_k = (x_{1,k}, x_{2,k})^T$ in terms of $\mathbf{x}_{k-1} = (x_{1,k-1}, x_{2,k-1})^T$, but the special (linear) structure of the equations will allow you to write down the solution of this system of equations by hand.

What is the situation now for large or small values of Δt? Does the solution *diverge* for large Δt? Is it accurate? **(20 points)**

Problem 3 (Numerical solution of a second-order ODE). A rocket that is shot up vertically experiences upward acceleration from its engines, and downward acceleration due to gravity. Its height therefore satisfies Newton's law

$$d''(t) = \frac{F(t)}{m(t)}, \tag{1}$$

where $d(t)$ denotes the distance from the earth's center. Assume that the rocket is initially at rest at $d(0) = 6371000$. After ignition, the engines produce a constant thrust for 10 minutes before shutting down:

$$T(t) = \begin{cases} 12 & \text{for } t < 600, \\ 0 & \text{for } t \geq 600. \end{cases}$$

On the other hand, gravity generates the force

$$G(t) = -(6371000)^2 \frac{10m(t)}{d(t)^2}.$$

(The factors here are chosen in such a way that at the surface – i.e., at $d = 6371000$ meters from the center of the Earth – the gravity equals 10 meters per second square, i.e., approximately the correct value. Furthermore, as is indeed the case, gravity decreases with the square of the distance.) The total force is then $F(t) = T(t) + G(t)$. The mass of the rocket decreases while fuel is burnt in the engines according to

$$m(t) = \begin{cases} 1 - \frac{0.9t}{600} & \text{for } t < 600, \\ 0.1 & \text{for } t \geq 600. \end{cases}$$

Rewrite this second order ordinary differential equation as a system of two first order equations. Then numerically approximate the altitude of the rocket for times between $t = 0$ and $t = 36000$ using the explicit Euler method. Try to determine the altitude at $t = 36000$ up to an accuracy of 1000 meters by playing with the size of the time step Δt. **(20 points)**