MATH 620: Variational Methods and Optimization I

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Lectures: Engineering E 206, Mondays/Wednesdays/Fridays, 12-12:50pm
Office hours: Wednesdays, 1-2pm; or by appointment.

Homework assignment 5 – due Friday 11/9/2018

Problem 1 (The dual space of a vector space X). The dual space X' of a vector space X is the set of all linear, continuous functionals $\varphi : X \to \mathbb{R}$.

We say that a functional φ is bounded if it satisfies the condition

$$|\varphi(x)| \le c\|x\|_X$$

for all $x \in X$ and with some constant $c = c(\varphi) < \infty$.

Show that (i) if $\varphi \in X'$, then it is bounded; and (ii) that if a linear functional is bounded, then it is also continuous. (In other words, a linear functional φ is in X' *if and only if* it is also bounded.) **(25 points)**

Problem 2 (The dual space of L^p). Every $a \in L^q$ induces a bounded linear functional $\varphi : L^p \to R$ of the form

$$\varphi(u) = \int_\Omega a(x)u(x)\,\mathrm{d}x.$$

Here and below, we will always assume that $\frac{1}{p} + \frac{1}{q} = 1$. Show in a first step that this functional is linear and bounded (and consequently continuous), i.e., that indeed we have $\varphi \in (L^p(\Omega))'$.

It is of course conceivable that X' is indeed larger than just the functionals introduced above. One of the possibilities would be that one could choose a larger class of functions a than just the L^q functions above. Show that this is not the case, i.e., that a function $a \notin L^q$ (for example if $a \in L^r$ with $r < q$ but $a \notin L^q$) does not induce a functional $\varphi \in X'$.

This statement is not easy to show in its full generality – though as often, it's all about finding the right approach. Here, it means showing that for such an a, φ can not be linear and continuous (or linear and bounded). If you don't see how to do this, create an example: Pick a particular p and corresponding q, then choose a specific $a \in L^r \backslash L^q$ (for example, a function with a singularity) and show that the corresponding φ is either not linear, not continuous, or not bounded by playing with functions $u \in L^p$ and investigating what $\varphi(u)$ is.

(It is worth noting that this argument only shows that linear functionals of the form shown above with $a \in L^r \backslash L^q$ do not give rise to functionals φ in $(L^p(\Omega))'$. It does not show that there are no *completely different* ways to construct linear and continuous functionals. It is the Riesz representation theorem that states that such other ways do not, in fact, exist.) **(25 points)**

Problem 3 (The norm on the dual space of L^p). We saw in class that every (dual) functional $\varphi \in (L^p(\Omega))'$ can be written in the form

$$\varphi(u) = \int_\Omega a(x)u(x)\,\mathrm{d}x$$

for some $a \in L^q$. Furthermore, we have defined the norm on the dual space as

$$\|\varphi\|_{(L^p(\Omega))'} = \sup_{u \in L^p(\Omega)} \frac{|\varphi(u)|}{\|u\|_{L^p(\Omega)}}.$$

Prove that

$$\|\varphi\|_{(L^p(\Omega))'} = \|a\|_{L^q(\Omega)}.$$

(Hint: It is not difficult to show that $\|\varphi\|_{(L^p(\Omega))'} \leq \|a\|_{L^q(\Omega)}$. To complete the proof, find a $u \in L^p$ so that $|\varphi(u)| = \|a\|_{L^q(\Omega)} \|u\|_{L^p(\Omega)}$ for which a good approach is to try $u(x) = |a(x)|^s \operatorname{sign}(a(x))$ with some exponent s to be chosen conveniently.) **(25 points)**

Problem 4 (Weak convergence). Define the function w on $[0,1]$ as

$$w(x) = \begin{cases} \alpha & \text{if } x \leq \frac{1}{2} \\ \beta & \text{if } x > \frac{1}{2}, \end{cases}$$

and let $\bar{w}$ be its periodic extension to all of $\mathbb{R}$.

Next consider the following sequence of functions on the set $\Omega = [0,1]$:

$$u_n(x) = \bar{w}(nx).$$

Show the following statements

(a) The sequence u_n does not converge strongly to any u in any of the L^p spaces, $1 \leq p \leq \infty$.

(b) We have

$$u_n(x) \rightharpoonup \frac{\alpha+\beta}{2} \qquad \text{in } L^p$$

for any $1 \leq p < \infty$.

(c) We have

$$u_n(x) \overset{*}{\rightharpoonup} \frac{\alpha+\beta}{2} \qquad \text{in } L^\infty.$$

(A similar result is actually true for more general cases: If you start with *any* bounded function $w : [0,1] \to \mathbb{R}$ and its periodic extension $\bar{w}$, then the u_n defined above converge weakly or weak-* to the *mean value* $\int_0^1 w(x)\,\mathrm{d}x$. The proof is not much different but does not shed any further light on the issue, so we are content with the simpler case above.) **(15 points)**

dacorogna, ex 1.2.3 on p. 19

show that strong implies weak convergence based on holdre ineq p 21 but not the other way around

ex 1.17 p. 22 ex1.18

ex1.19

1.20iv

ex 1.3.1, p 27, by comparing with the finite dimensional case