MATH 620: Variational Methods and Optimization I

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Lectures: Engineering E 206, Mondays/Wednesdays/Fridays, 12-12:50pm
Office hours: Wednesdays, 1-2pm; or by appointment.

Homework assignment 4 – due Friday 10/26/2018

Problem 1 (Equivalence of norms on $\mathbb{R}^n$). On finite dimensional spaces, all norms are equivalent. Let us test this for a subset of all possible norms on $\mathbb{R}^n$: Show that there exist constants $0 < c_{12} \le C_{12} < \infty$ so that

$$c_{12}\|x\|_{l_1} \le \|x\|_{l_2} \le C_{12}\|x\|_{l_1}$$

for all vectors $x \in \mathbb{R}^n$. In other words, that the l_1 and l_2 norms on $\mathbb{R}^n$ are "equivalent".

Then show the same for the l_2 and l_∞ norms: Show that there exist constants $0 < c_{2\infty} \le C_{2\infty} < \infty$ so that

$$c_{2\infty}\|x\|_{l_2} \le \|x\|_{l_\infty} \le C_{2\infty}\|x\|_{l_2}$$

for all vectors $x \in \mathbb{R}^n$.

Combine these estimates to show that the l_1 and the l_∞ norms are equivalent. **(20 points)**

Problem 2 (Equivalence of norms on the space of polynomials). Consider the ($n+1$ dimensional) space of polynomials of degree at most n:

$$X := \left\{ u : [0,1] \to \mathbb{R} \ : u = \sum_{k=0}^{n} a_n x^n \right\}.$$

Define on this space the following norms:

- $\|u\|_1 = \max_{x\in[0,1]} |u(x)|$,
- $\|u\|_2 = \max_{x\in[0,1]} |u(x)| + \max_{x\in[0,1]} |u'(x)|$,
- $\|u\|_3 = \left(\int_0^1 |u(x)|^2 \, dx\right)^{1/2}$.

(You will notice that these are the L^∞, $W^{1,\infty}$, and L^2 norms when taking into account that the members of X are all functions that are continuous and continuously differentiable.) Then answer the following questions:

(a) For each of these three norms, show that it is indeed a norm, i.e., that it satisfies the norm axioms.

(b) Show that all of these norms are equivalent on X – as they ought to be because the space X is finite dimensional.

(20 points)

Problem 3 (Equivalence of norms on infinite dimensional spaces). Consider the (infinite dimensional) vector space $C^1((0,1))$. Define on this space the same three norms as before:

- $\|u\|_1 = \max_{x\in[0,1]} |u(x)|$,
- $\|u\|_2 = \max_{x\in[0,1]} |u(x)| + \max_{x\in[0,1]} |u'(x)|$,
- $\|u\|_3 = \int_0^1 |u(x)|^2 \, dx$.

Then answer the following questions:

(a) For each of these three norms, show that it is indeed a norm on X, i.e., that it satisfies the norm axioms. If you have shown this in sufficient generality in the previous problem, you can just refer to that solution; otherwise, you will have to generalize the arguments to the larger space C^1.

(b) Are any of these norms equivalent to each other? If you can't show that they are, demonstrate that they are *not* equivalent by showing that no constants $0 < c \le C < \infty$ can exist as are necessary for equivalence of norms.

(20 points)

Problem 4 (Membership in $W^{k,p}$). Think about singular functions $u : B_1(0) \subset \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ of the form

$$u(x) = \frac{1}{\|x\|^s}$$

with $s > 0$.

For given values of the space dimension $d \ge 1$, the degree $k \ge 0$, and the exponent $1 \le p \le \infty$, state for which values s the function satisfies $u \in W^{k,p}(B_1(0))$.

The spaces $H^k = W^{k,2}$ have special importance in the theory of partial differential equations. Does the space $H^1 = W^{1,2}$ contain any singular functions with $s > 0$ for $d = 1$? For $d = 2$? For $d > 3$? How about the space $H^2 = W^{2,2}$? **(20 points)**

Problem 5 (Weak derivatives and membership in $W^{1,2}$). We saw in class that the discontinuous function of one argument,

$$u(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x \ge 0 \end{cases}$$

does not have a weak derivative.

But how about the function $u : \mathbb{R}^2 \to \mathbb{R}$ of two arguments,

$$u(\mathbf{x}) = \sin(\arctan(x_2/x_1))$$

that is discontinuous at the origin? Can you guess a weak gradient for this function u (which is of course a two-dimensional vector field) and prove that it really is *the* weak gradient?

If so, what spaces $W^{1,p}(B_1(0))$ is u in if we restrict it to the unit ball in $\mathbb{R}^2$?

(Hint: Plot the function. Then think about whether there is possibly a coordinate system better suitable to the task.) **(20 points)**