MATH 620: Variational Methods and Optimization I

Instructor: Prof. Wolfgang Bangerth
Weber 214
bangerth@colostate.edu

Lectures: Engineering E 206, Mondays/Wednesdays/Fridays, 12-12:50pm
Office hours: Wednesdays, 1-2pm; or by appointment.

Homework assignment 2 – due Friday 9/28/2018

Problem 1 (Affine subspaces, convexity). Consider the vector space $X = C^k([0,1])$ of functions that are k times continuously differentiable. We will often have to impose boundary conditions, so take the subset

$$D = \left\{ \varphi \in X : \varphi(0) = 2, \frac{\partial \varphi}{\partial x}(1) = 3 \right\} \subset X$$

of functions with a prescribed function value on the left, and prescribed derivative on the right end of the interval. (This clearly only makes sense if $k \geq 1$.)

(a) Show that this set is an affine subspace of X.

(b) Show that D is a convex set.

(a) State what the tangent space D' of D is.

(10 points)

Problem 2 (Directional derivatives). Take the function

$$f(\vec{x}) = \min\{|x_1|, |x_2|\}\text{sign}(x_1).$$

In class, we talked about the fact that the directional (Gateaux) derivative satisfies

$$Df(\vec{x}; \vec{v}) = (\nabla f(\vec{x})) \cdot \vec{v}.$$

For the current function, at the origin $\vec{x} = 0$, we have $\nabla f(0) = 0$, but the directional derivative is not zero for all $\vec{v}$. Explain the discrepancy. What does this imply for the viability of the idea that we can look for points with $\nabla f = 0$ when searching for minima of functions?

(10 points)

Problem 3 (Directional derivatives). Take the following variation of the function of the previous problem:

$$f(\vec{x}) = \min\{|x_1|, |x_2|\}.$$

Show that this function still has $\nabla f(0) = 0$, but that it does not have a Gateaux derivative $Df(\vec{x}; \vec{v})$ for all $\vec{v}$.

(10 points)

Problem 4 (Subdifferentials). Take the functions

$$
\begin{aligned}
f_1(\vec{x}) &= |x_1| + |x_2|, \\
f_2(\vec{x}) &= \sqrt{|x_1|^2 + |x_2|^2}, \\
f_\infty(\vec{x}) &= \max\{|x_1|, |x_2|\}.
\end{aligned}
$$

For each, of these, (a) determine the subdifferential $\partial f(\vec{x})$ for each $\vec{x} \in \mathbb{R}^2$; (b) plot $\partial f(0)$ when evaluated at the origin $\vec{x} = 0$; (c) verify that the necessary condition for minima $0 \in \partial f(\vec{x})$ is only satisfied at the origin.

(20 points)