MATH 545: Partial Differential Equations I

Instructor:	Prof. Wolfgang Bangerth Weber 214 bangerth@colostate.edu
Lectures:	Engineering E 206, Mondays/Wednesdays/Fridays, 10-10:50am
Office hours:	Wednesdays, 1-2pm; or by appointment.

Homework assignment 6 – due Friday 11/30/2018

Problem 1 (The wave equation via a Fourier series). We have seen two different ways in class how to solve the one-dimensional wave equation. Consider the following setting on the domain $\Omega = (0, 4)$:

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} u(x,t) - c^2 \frac{\partial^2}{\partial x^2} u(x,t) &= 0 && \text{for all } x \in \Omega,\ t > 0, \\
u(x,0) &= h(x) && \text{for all } x \in \Omega, \\
\frac{\partial}{\partial t} u(x,0) &= 0 && \text{for all } x \in \Omega, \\
u(0,t) &= 0 && \text{for all } t > 0, \\
u(4,t) &= 0 && \text{for all } t > 0,
\end{aligned}$$

where we will choose the initial displacement $h(x)$ as

$$h(x) = \begin{cases} 1 & \text{if } x < 1, \\ 0 & \text{of } 1 \le x \le 4. \end{cases}$$

Compute the solution of this problem via the Fourier series representation we have derived in class, i.e., find the coefficients A_n, B_n so that

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right) \sin \frac{n\pi x}{L},$$

where $\omega_n = \frac{n\pi c}{L}$ and so that $u(x,t)$ solves the problem above. **(20 points)**

Problem 2 (The wave equation via a Fourier series). For the solution you have found above, create computer-generated plots of $u_N(x,t)$ at $t = 1, 2, 6, 10, 20$ and for $N = 10, 50, 100$, where u_N is the solution that takes into account only the first N Fourier terms. That is,

$$u_N(x,t) = \sum_{n=1}^{N} \left(A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right) \sin\left(\frac{n\pi x}{L}\right),$$

with A_n, B_n computed in Problem 1. For simplicity, use $c = 1$. (But you can also play with different values for c to see how that affects the solution.) **(20 points)**

Problem 3 (The wave equation via d'Alembert's solution). Take the same problem again and solve it via the approach by d'Alembert. That is, create the necessary extension $\tilde{u}_0$ to all of $\mathbb{R}$ of the initial condition u_0 that is only defined on $\Omega = (0, 4)$. Create computer-generated plots of the solution $u(x,t)$ you get from $\tilde{u}_0$ at times $t = 1, 2, 6, 10, 20$ on Ω. Again, use $c = 1$ or another value of your choice (but so that you can compare with the solution plotted in the previous problem). **(20 points)**

Problem 4 (Wave equation). The total kinetic and potential energy of a vibrating string of length L at any given time t is

$$E(t) = \frac{1}{2}\int_0^L \left(\frac{\partial u(x,t)}{\partial t}\right)^2 + c^2 \left(\frac{\partial u(x,t)}{\partial x}\right)^2 \, dx.$$

Show that the energy is conserved, i.e. that $E(t_1) = E(t_2)$ for any two time instants t_1, t_2 if $u(x,t)$ satisfies the homogenous wave equation

$$\begin{aligned} \frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} &= 0, && \text{in } (0,L)\times(0,T), \\ u(0,t) &= 0 && \text{for } t \in [0,T], \\ u(L,t) &= 0 && \text{for } t \in [0,T], \end{aligned}$$

plus initial conditions that can be arbitrary here.

Hint: First note that $E(t_2) - E(t_1) = \int_{t_1}^{t_2} \frac{\partial}{\partial t} E(t) \; dt$. Then derive what form $\frac{\partial}{\partial t}E(t)$ has by direct differentiation under the integral in the definition of $E(t)$. Then integrate by parts in space and time as necessary and see if you can cancel terms using the wave equation and its boundary values as stated above.

(10 points)

Problem 5 (Self-adjoint operators). We say that a (real) matrix $A \in \mathbb{R}^{n\times n}$ is self-adjoint if $x^T(Ay) = (Ax)^T x$ for all vectors $x, y \in \mathbb{R}^n$. If we write the scalar product between two vectors as $\langle x, y\rangle = x^T y = x \cdot y$, then this means that A is self-adjoint if $\langle x, Ay\rangle = \langle Ax, y\rangle$. For real matrices this implies that A is self-adjoint if and only if it is symmetric. (If the matrix had complex entries, the condition would be that A needs to be "Hermitian".)

For operators, the situation is essentially similar. Ignoring minor subtleties, we say that an operator L acting on functions is self-adjoint if

$$\int_\Omega v(x)\,(Lw(x))\,dx = \int_\Omega (Lv(x))\,w(x)dx$$

for all functions v, w in the domain of L. An important aspect to take into account for operators is that we need to carefully specify this domain of the operator.

Show that the following operators are indeed self-adjoint:

- $L = -\Delta$ on a domain $\Omega \subset \mathbb{R}^n$ when applied to functions $v, w : \Omega \to \mathbb{R}$ that satisfy $v|_{\partial\Omega} = 0$ and similarly for w.
- $L = -\Delta^2$ on a domain $\Omega \subset \mathbb{R}^n$ when applied to functions $v, w : \Omega \to \mathbb{R}$ that satisfy $v|_{\partial\Omega} = 0$, $\mathbf{n}\cdot\nabla v|_{\partial\Omega} = 0$ and similarly for w.
- $L = -\nabla\cdot(A\nabla)$ where $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix, $\Omega \subset \mathbb{R}^n$, and where L is applied to functions $v, w : \Omega \to \mathbb{R}$ that satisfy $v|_{\partial\Omega} = 0$ and similarly for w.

Also show that the operator $L = \mathbf{b}\cdot\nabla$ with a fixed vector $\mathbf{b} \in \mathbb{R}^n$ (i.e., the derivative *in direction of* $\mathbf{b}$) is in general not self-adjoint.

In all cases where this is necessary, you can assume that functions are smooth enough so that integration by parts is allowed.

(10 points)

Problem 6 (Eigenfunctions on the square). We have seen in class that the eigenfunctions and eigenvalues of the Laplacian are quite useful in many contexts. Find those functions $\phi(x, y)$ that satisfy

$$\begin{aligned} -\Delta\phi(x, y) &= \lambda\phi(x, y), && \text{in } \Omega, \\ u(x, y) &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where we choose $\Omega = (0, L) \times (0, H)$, i.e., a rectangle. Because of the special structure of the domain, you may want to find these functions by making the assumption that they can be written as $\phi(x, y) = X(x)Y(y)$, and then work out what $X(x)$ and $Y(y)$ must be.

Let us assume that you have found a number of such functions and that you somehow indexed them as ϕ_k, $k = 1, 2, 3, \ldots$. Similarly, let us call the corresponding eigenvalues λ_k. Then verify the claims we have made in class for this special domain, namely:

- There are infinitely many such eigenfunction/eigenvector pairs.
- All eigenvalues are positive.
- The eigenvalues grow without bound, i.e. $\lambda_k \to \infty$ as k grows.
- The eigenfunctions are mutually orthogonal, i.e.,

  $$\int_\Omega \phi_n(x, y)\phi_m(x, y)\, \mathrm{d}x\, \mathrm{d}y = 0$$

  for $n \neq m$. (Because we can multiply any eigenfunction by a scalar and it remains an eigenfunction, it is then possible to *normalize* them so that in addition to the orthogonality condition above, we also have $\int_\Omega \phi_n(x, y)^2\, \mathrm{d}x\, \mathrm{d}y = 1$.)

(20 points)