MATH 545: Partial Differential Equations I

Instructor: Prof. Wolfgang Bangerth
Weber 214
bangerth@colostate.edu

Lectures: Engineering E 206, Mondays/Wednesdays/Fridays, 10-10:50am
Office hours: Wednesdays, 1-2pm; or by appointment.

Homework assignment 5 – due Friday 11/9/2018

Problem 1 (Solution of the heat equation on a rectangle). If the domain on which we want to solve the heat equation is a rectangle, i.e., $\Omega = (0, L) \times (0, H) \subset \mathbb{R}^2$, then we had derived that the solution of the heat equation with zero boundary values,

$$\begin{aligned} \frac{\partial}{\partial t} u(x, y, t) - k\Delta u(x, y, t) &= 0 \qquad \text{for all } (x, y) \in \Omega,\ t > 0, \\ u(x, y, t) &= 0 \qquad \text{for all } (x, y) \in \partial\Omega,\ t > 0, \end{aligned}$$

has the general form of a superposition of solutions we have found through the separation of variables approach. Namely, that the solution has the form

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} e^{-\left(\frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{H^2}\right)t} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right)$$

Compute what values the coefficients A_{nm} must take on so that the solution $u(x, y, t)$ satisfies the following initial conditions:

$$u(x, y, 0) = u_0(x, y) = \begin{cases} 0 & \text{if } x < \frac{L}{2} \text{ and } y < \frac{H}{2}, \text{ or if } x \geq \frac{L}{2} \text{ and } y \geq \frac{H}{2}, \\ 1 & \text{otherwise.} \end{cases}$$

(20 points)

Problem 2 (Plotting 2d solutions). Given what you found in the previous problem, show a computer plot of $u(x, y, t)$ for $t = 0$ (and verify that it matches the initial conditions) as well as for $t = 0.1$, $t = 0.5$, $t = 1$, $t = 10$.

Use $L = 2, H = 1, k = 1$ for the constants that appear in the problem. **(20 points)**

Problem 3 (Fourier series). Derive the Fourier series on $[-\pi, \pi]$ of the function $f(x) = x$. From this series, derive the Fourier series of $F(x) = x^2/2$ without using the formulas $\frac{1}{L}\int_{-L}^{L} F(x) \cos nx \, dx$ (and similar for the sine terms) to compute the coefficients A_0, A_n, B_n of the second series. **(10 points)**

Problem 4 (Fourier series). Derive the Fourier series on $[-\pi, \pi]$ of the function $F(x) = x$ (it is the same as in the previous problem). State whether you can derive the Fourier series of the function $f(x) = 1$ from it by differentiating each term in the series of $F(x)$ individually. State the Fourier series of $f(x)$.

(10 points)

Problem 5 (Fourier series). Generate computer plots of the partial sums $\bar{f}_{10}(x) = a_0 + \sum_{n=1}^{10} a_n \cos(nx) + \sum_{n=1}^{10} b_n \sin(nx)$ and $\bar{F}_{10}(x) = A_0 + \sum_{n=1}^{10} A_n \cos(nx) + \sum_{n=1}^{10} B_n \sin(nx)$ consisting of the first 10 terms for the Fourier series of the functions $f(x) = 1, F(x) = x$, where the Fourier series is calculated over the interval $-\pi \ldots \pi$. Plot these Fourier series on the larger interval $-2\pi \ldots 2\pi$.

Also generate a plot of the partial sum $g_{10}(x) = \sum_{n=1}^{10} -A_n n \sin(nx) + \sum_{n=1}^{10} B_n n \cos(nx)$, using the coefficients A_n, B_n of the Fourier series of $F(x) = x$ (this is the term-by-term differentiated Fourier series of $F(x)$ for which one may hope that it matches $f(x)$.) Repeat this last part with 20, 30, 50, 100 terms instead of just 10. Do you think this series converges?

(20 points)

Problem 6 (Fourier series). Calculate or look up the Fourier series on the interval $-\pi \ldots \pi$ of the function

$$f(x) = \begin{cases} 1 \text{ for } x > 0 \\ -1 \text{ for } x \leq 0 \end{cases}$$

Generate plots of the partial sums $f_N(x) = A_0 + \sum_{n=1}^{N} A_n \cos(nx) + \sum_{n=1}^{N} B_n \sin(nx)$ consisting of the first N terms, for each value $N = 2, 5, 10, 20, 50$. Try to determine the maximal difference $|f(x) - f_N(x)|$ numerically or graphically. We know that for $N \to \infty$, $f_N(x) \to f(x)$ almost everywhere; is this consistent with your results?

(20 points)