MATH 545: Partial Differential Equations I

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Lectures:	Engineering E 206, Mondays/Wednesdays/Fridays, 10-10:50am
Office hours:	Wednesdays, 1-2pm; or by appointment.

Homework assignment 1 – due Monday 9/17/2018

Problem 1 (Bivariate analysis). Here is a picture of the large radio telescope in Arecibo, Puerto Rico:

Impose a coordinate system with the origin at the center of the dish and such that the positive x-axis runs from the origin in the direction of the tower in front. Let Ω be the domain in x-y-space occupied by the dish. Let $H(x,y)$ be the height of the telescope's surface above the level defined by the circular rim (the surface is of course below the rim, so $H(x,y) \le 0$).

a) Plot the coordinate system (i.e. x- and y-axes) into the picture. Indicate $H(0,0)$.

b) Describe in words the meaning of the following quantities defined on the entire domain:

$$\frac{\partial H(x,y)}{\partial x} \qquad \frac{\partial H(x,y)}{\partial y} \qquad \nabla H(x,y)$$

$$\int_\Omega H(x,y)\,dx\,dy \qquad \int_{-R}^{R} H(x,0)\,dx \qquad \nabla H(0,0)$$

$$\frac{\partial^2 H(x,y)}{\partial x^2} \qquad \frac{\partial^2 H(x,y)}{\partial y^2} \qquad \Delta H(x,y)$$

For the terms on the last line, also state the sign for the function $H(x, y)$.

c) Describe in words the meaning of the following quantities defined on the boundary of the domain and state the sign of quantities where possible:

$$\mathbf{n} \qquad \frac{\partial H(x,y)}{\partial n} = \mathbf{n} \cdot \nabla \partial H(x,y) \qquad \frac{\partial^2 H(x,y)}{\partial n^2}$$

$$\int_{\partial\Omega} H(x,y)\, ds \qquad \int_{\partial\Omega} \frac{\partial H(x,y)}{\partial n}\, ds$$

(20 points)

Problem 2 (Integration by parts 1). Calculate the following integrals using integration by parts:

a) $\int_0^\pi x \sin x \, dx$

b) $\int_0^1 x e^x \, dx$

c) $\int_0^1 x^3 e^x \, dx$

(15 points)

Problem 3 (Integration by parts 2). Let $f(x, y) = x^2 + y^2$ and $g(x, y) = \sin(xy)$. State which of the following statements is true and why or why not:

a) $\int_{-\pi}^{\pi} f(x, 0) g(x, 0) \, dx = 0$

b) for every y there holds

$$\int_{-1}^{1} \frac{\partial f(x,y)}{\partial x} g(x,y)\, dx \quad = \quad -\int_{-1}^{1} \frac{\partial g(x,y)}{\partial x} f(x,y)\, dx \quad + \quad f(1,y)g(1,y) \quad - \quad f(-1,y)g(-1,y)$$

c) for every y there holds (note the signs)

$$\int_{-1}^{1} \frac{\partial f(x,y)}{\partial x} g(x,y)\, dx \quad = \quad +\int_{-1}^{1} \frac{\partial g(x,y)}{\partial x} f(x,y)\, dx \quad - \quad f(1,y)g(1,y) \quad + \quad f(-1,y)g(-1,y)$$

d) for every x there holds

$$\int_{-1}^{1} \frac{\partial f(x,y)}{\partial x} g(x,y)\, dy \quad = \quad -\int_{-1}^{1} \frac{\partial g(x,y)}{\partial x} f(x,y)\, dy \quad + \quad f(x,1)g(x,1) \quad - \quad f(x,-1)g(x,-1)$$

(10 points)

Problem 4 (Divergence theorem). For the simple case of the unit square $\Omega = [0, 1]^2$, show that the divergence theorem

$$\int_\Omega \operatorname{div} \mathbf{u} \, dx\, dy = \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{u} \, dl$$

holds for all sufficiently smooth vector fields $\mathbf{u}$. Hint: Use (i) that the integral over Ω is really a double integral over $0 \leq x, y \leq 1$, (ii) that the expression div $\mathbf{u}$ has two terms, (iii) that each of these expressions can individually be integrated by parts in either the x or y direction, and (iv) that in the surface integral on the right you can express the normal vector explicitly on each part of the boundary. **(20 points)**

Problem 5 (More integration by parts). You have seen the general form of the higher dimensional integration by parts theorem: For a (sufficiently smooth) vector field $\vec{\phi}(\mathbf{x})$ and a scalar field $u(\mathbf{x})$ both defined on a domain $\Omega \subset \mathbb{R}^n$, the following is true:

$$\int_\Omega u(\mathbf{x}) \text{div } \vec{\phi}(\mathbf{x}) \, dx = -\int_\Omega \nabla u(\mathbf{x}) \cdot \vec{\phi}(\mathbf{x}) \, dx + \int_{\partial\Omega} u(\mathbf{x})\vec{\phi}(\mathbf{x}) \cdot \vec{n}(\mathbf{x}) \, dx.$$

This identity is true for any space dimension n, so in particular it must be true for $n = 1$.

Argue how you can derive the identity

$$\int_a^b u'v \; dx = -\int_a^b uv' \; dx + u(b)v(b) - u(a)v(a)$$

from it by thinking about what the divergence, the gradient, the vector field $\vec{\phi}$, and the normal vector $\vec{n}$ look like in the 1d case. **(10 points)**

Problem 6 (Solutions of the Laplace equation). The steady state temperature in a one-dimensional object without heat sources is described by the following equation, as derived in class:

$$-\frac{d}{dx}\left[k(x)\frac{d}{dx}T(x)\right] = 0,$$

for all $x \in \Omega \subset \mathbb{R}$, and where $k(x)$ is the heat conductivity coefficient. This equation needs to be augmented by appropriate boundary conditions.

Solve this equation analytically for the case where $\Omega = (0, L)$ with $L = 10$, $T(0) = 1$, $T(L) = 2$, and for the following two cases:

- $k(x) = 1$, i.e., the one-dimensional object is homogenous,
- $k(x) = 1$ for $x < L/2$ and $k(x) = 2$ for $x \geq L/2$.

In both cases plot the solution and interpret it. **(25 points)**