MATH 561: Numerical Analysis I

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Homework assignment 4 – due 3/21/2017

Problem 1 (Conjugate Gradient iteration). Repeat problem 5 of Homework 3 one more time, but with the Conjugate Gradient method for solving linear problems. Update your assessment on how quickly all of the methods you have implemented converge for the problem at hand. **(20 points)**

Problem 2 (Bisection method by hand). A ball is dropped from a tower. During its fall it is subject to the forces of gravity and air resistance. Thus, the height h in ft as a function of time t in s is given by the formula

$$h(t) = h_0 - \frac{mg}{k}t + \frac{m^2 g}{k^2}\left(1 - e^{-\frac{kt}{m}}\right).$$

Here, $h_0 = 300\,\text{ft}$ is the height of the tower, $m = 1/2\,\text{lb}$ the mass of the ball, $g = 32.17\,\text{ft/s}^2$ is the gravitation of earth and $k = 1/4\,\text{lb/s}$ is the air resistance coefficient.

Use the bisection method (on paper, with pen to write down the steps, and a calculator to evaluate the formula above) to find out when the ball hits the ground with an accuracy of 0.1 seconds. **(15 points)**

Problem 3 (Bisection method with a computer). Implement a program that solves the same problem as above on a computer. Assess how many iterates you need to compute the time at which the ball hits the ground to an accuracy of 10^{-2} seconds, 10^{-4} seconds, 10^{-6} seconds, 10^{-8} seconds, and 10^{-12} seconds, respectively. **(15 points)**

Problem 4 (Newton's method). For certain functions, Newton's method will always converge in a single step, no matter where we start. What functions are these, and why is a single step enough? (Hint: Think about the graphical interpretation of Newton's method, and when it will produce a new iteration that falls exactly onto the true root of the function.) **(5 points)**

Problem 5 (Newton's method). For functions $f(x)$ of one variable x, Newton's method almost always converges very quickly (in a matter of a few iterations). However, almost always is not always, and we can find examples where Newton's method converges rather slowly.

Write a program to find the zero $x = 1$ of the function

$$f(x) = x^{25} - 1$$

that uses Newton's method and starts at $x_0 = 20$.

(a) How many iterations do you need to achieve an accuracy of 10^{-8}?

(b) You will observe very slow convergence. Can you explain from the formulas that express the error e_n as a function of e_{n-1} why convergence is so slow?

(c) Does the method still converge of second order, i.e. is the relationship between e_n and e_{n+1} derived in class true also for this problem?

(d) What answers do you get to the questions in (a)–(c) if you apply the same program to the function $f(x) = x^3 - 1$ instead, again starting from $x_0 = 20$? **(25 points)**

Problem 6 (Convergence order for sequences). Determine the order of convergence and the asymptotic error constant for the following sequences:

(a)

$$a_n = 5.0625, 2.25, 1, \frac{4}{9}, \frac{16}{81}$$

(b)

$$b_n = 2.718, 2.175, 1.740, 1.392, 1.113, 0.8907$$

(c)

$$c_n = 0.318, 0.180, 0.0761, 0.021, 3.04 \cdot 10^{-3}, 1.68 \cdot 10^{-4}, 2.17 \cdot 10^{-6}$$

(c)

$$d_n = 0.9, 0.899473, 0.898894, 0.898257, 0.897557, 0.896787, 0.895942, \\ 0.895013, 0.893992, 0.89287, 0.891638, 0.890284, 0.888798.$$

All of these sequences converge to zero, though at different speeds. For each sequence, determine how many terms are necessary to get the value below 10^{-8}, i.e., determine the smallest n so that $a_n \leq 10^{-8}$ (and similarly for b_n, c_n, d_n). **(20 points)**

Happy spring break!