MATH 561: Numerical Analysis I

Instructor: Prof. Wolfgang Bangerth
bangerth@colostate.edu

Homework assignment 2 – due 2/14/2017

Problem 1 (LU decomposition). Solve the linear system $Ax = b$ with the Hilbert matrix system we already saw in Problem 2:

$$A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

by applying the following steps with paper and pencil:

1. Compute the LU decomposition of A and write down the elimination steps.
2. Use forward and backward substitution to obtain the solution x.

(10 points)

Problem 2 (LU decomposition). Write a program that implements the LU decomposition algorithm for general $n \times n$ matrices and outputs the L and U factors. Apply it to the matrix of the linear system

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

In a second step, implement the backward and forward substitution solves with the upper and lower triangular factors L and U for any given vector. Apply it to the right hand side above and verify that your solution is correct.

(15 points)

Problem 3 (Positive definite matrices). Positive definite matrices are those matrices for which $x^T Ax > 0$ for all vectors $x \neq 0$. These matrices play an important role in many applications of engineering and physics. Let us consider one of their properties.

Any matrix A can be written as $A = A^s + A^a$, where the symmetric part A^s and the skew-symmetric part A^a of a matrix are defined as

$$A^s = \frac{A + A^T}{2}, \qquad A^a = \frac{A - A^T}{2}.$$

Prove that if A is positive definite then A^s is positive definite, and vice versa.

(5 points)

Problem 4 (Norms on $\mathbb{R}^n$). In the analysis of iterative solution methods for linear systems, we often come across different vector norms. A functional $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is called a *norm* if it satisfies the following three conditions:

1. $\|x\| \geq 0$ for all vectors $x \in \mathbb{R}^n$ and $\|x\| = 0$ if and only if $x = 0$ (positive definiteness);
2. $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{R}$ and all vectors $x \in \mathbb{R}^n$ (scalability);
3. $\|x + y\| \leq \|x\| + \|y\|$ for all vectors $x, y \in \mathbb{R}^n$ (triangle inequality).

Determine which of the following are norms on $\mathbb{R}^n$ by proving or disproving that they satisfy the three conditions above:

a) $\max_{1 \leq i \leq n} |x_i|$

b) $\max_{2 \leq i \leq n} |x_i|$

c) $\sum_{i=1}^n |x_i|^3$

d) $\left(\sum_{i=1}^n |x_i|^{1/2}\right)^2$

e) $\max\{|x_1 - x_2|, |x_1 + x_2|, |x_3|, |x_4|, \ldots, |x_n|\}$

f) $\sum_{i=1}^n 2^{-i} |x_i|$ **(12 points)**

Problem 5 (Equivalence of norms on $\mathbb{R}^n$). In class, we proved the equivalence of the norms $\|.\|_\infty$ and $\|.\|_2$. Here now, you are asked to prove the same for $\|.\|_\infty$ and $\|.\|_1$.

a) Prove that there are indeed constants c, C such that

$$c\|v\|_\infty \leq \|v\|_1 \leq C\|v\|_\infty.$$

where

$$\|v\|_1 = \sum_i |v_i|,$$

$$\|v\|_\infty = \max_i |v_i|,$$

and where v is an n-dimensional vector in $\mathbb{R}^n$.

b) For vectors v_1, v_2 with $\|v_1\|_1 \leq \|v_2\|_1$, does the result of part a) imply that $\|v_1\|_\infty \leq \|v_2\|_\infty$? If not, give an example of vectors for which this does not follow. **(8 points)**

Problem 6 (Alternative vector norms). Let A be a symmetric and positive definite $n \times n$ matrix. Show that

$$\|x\|_A = \sqrt{x^T A x}$$

is a norm for vectors $x \in \mathbb{R}^n$. (Hint: Use the eigenvalue and eigenvector decomposition of symmetric positive definite matrices.)

(10 points)

Problem 7 (Jacobi iteration). Let A, b be the 100×100 matrix and 100-dimensional vector defined by

$$A_{ij} = \begin{cases} 2.01 & \text{if } i = j, \\ -1 & \text{if } i = j \pm 1, \\ 0 & \text{otherwise,} \end{cases} \qquad\qquad b_i = \frac{1}{100} \sin\left(\frac{2\pi i}{50}\right).$$

Apply Jacobi's method to solving $Ax = b$. Write a program that implements the Jacobi method and start with a vector x_0 with randomly chosen elements in the range $-1 \le (x_0)_i \le 1$ (i.e. with elements generated from what the `rand()` function or a similar replacement returns).

(Hint: It is not necessary to actually store the complete matrix just to multiply with it. Rather, use that the i-th component of the vector Ay is $(Ay)_i = \sum_{j=1}^{n} A_{ij} y_j = 2.01 y_i - y_{i-1} - y_{i+1}$ at least for $2 \le i \le n-1$, and obvious modifications for $j = 1$ and $j = n$.)

Run 200 Jacobi iterations and plot the values of $(x^{(k)})_i$ against i for every few iterations, for example $k = 0, 2, 5, 10, 20, 50, 100, 200$. What do you observe?

(20 points)

Problem 8 (Gauss-Seidel iteration). Repeat the previous problem, but use the Gauss-Seidel iteration instead to compute the vectors $x^{(k)}$. Generate the same plots as before. Compare your results with the previous results.

(20 points)