MATH 651: Numerical Analysis II

Instructor: Prof. Wolfgang Bangerth
bangerth@colostate.edu

Homework assignment 4 – due 11/06/2017

Problem 1 (Numerical solution of a ODE). Consider the following scalar ordinary differential equation (ODE):

$$x'(t) = \frac{1}{2}x(t), \qquad\qquad x(0) = 1.$$

The solution of this equation is $x(t) = e^{\frac{1}{2}t}$. Compute approximations to $x(4)$ using the

- first order Taylor expansion method,
- second order Taylor expansion method,
- implicit Euler method,
- trapezoidal (Crank-Nicolson) method,

each with step sizes $\Delta t = 2, 1, \frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{32}$. Compute their respective errors $e = |x_N - x(4)|$ where x_N is the approximation to $x(4)$ at the end of the last time step, and compute the convergence rates. Compare the accuracy of all these methods for the same step size Δt. **(20 points)**

Problem 2 (Numerical solution of a second-order ODE). A rocket that is shot up vertically experiences upward acceleration from its engines, and downward acceleration due to gravity. Its height therefore satisfies Newton's law

$$d''(t) = \frac{F(t)}{m(t)}, \tag{1}$$

where $d(t)$ denotes the distance from the earth's center. Assume that the rocket is initially at rest at $d(0) = 6371000$. After ignition, the engines produce a constant thrust for 10 minutes before shutting down:

$$T(t) = \begin{cases} 12 & \text{for } t < 600, \\ 0 & \text{for } t \geq 600. \end{cases}$$

On the other hand, gravity generates the force

$$G(t) = -(6371000)^2 \frac{10m(t)}{d(t)^2}.$$

(The factors here are chosen in such a way that at the surface – i.e., at $d = 6371000$ meters from the center of the Earth – the gravity equals 10 meters per second square, i.e., approximately the correct value. Furthermore, as is indeed the case, gravity decreases with the square of the distance.) The total force is then $F(t) = T(t) + G(t)$. The mass of the rocket decreases while fuel is burnt in the engines according to

$$m(t) = \begin{cases} 1 - \frac{0.9t}{600} & \text{for } t < 600, \\ 0.1 & \text{for } t \geq 600. \end{cases}$$

Rewrite this second order ordinary differential equation as a system of two first order equations. Then numerically approximate the altitude of the rocket for times between $t = 0$ and $t = 36000$ using the explicit Euler method. Try to determine the altitude at $t = 36000$ up to an accuracy of 1000 meters. **(30 points)**

Problem 3 (More rocketry). Let's explore the world of rocketry some more. Let's say you already have your satellite in orbit around Earth. At time t (measured in seconds), its three-dimensional position (measured in meters) is $\mathbf{x}(t)$. Then Newton tells us that

$$\mathbf{x}''(t) = \frac{\mathbf{F}(\mathbf{x}(t))}{m}, \tag{2}$$

where we here consider the mass as constant. On the other hand, the forces $\mathbf{F}$ are only due to gravity. Namely, if we place the Earth at the origin of the coordinate system, then

$$\mathbf{F}(\mathbf{x}(t)) = -(6371000)^2 \frac{10m}{|\mathbf{x}(t)|^2} \frac{\mathbf{x}(t)}{|\mathbf{x}(t)|}. \tag{3}$$

Find online the current altitude and speed of the International Space Station. Then convert this into an initial position vector $\mathbf{x}_0$ and initial velocity $\dot{\mathbf{x}}_0$; you can choose $\mathbf{x}_0$ as any point you want that has the correct altitude (but remember that $\mathbf{x}$ measures distance from the Earth center!) and $\dot{\mathbf{x}}_0$ as a vector that is perpendicular to $\mathbf{x}_0$ (i.e., it is tangential to the Earth surface).

Using this model, the orbit of the International Space Station should be roughly circular. In practice, the orbit will be slightly elliptic because you may have gotten the altitude or velocity slightly wrong by a few percent. Either way, some elementary physical considerations show that the orbit must be closed, i.e., that the orbiting object returns to the same location with the same velocity after some time, and then keeps doing this over and over again. However, numerical approximations may not satisfy this property.

To test this assertion, run each of the following methods on this model:

- the explicit Euler method,
- the implicit Euler method,
- the trapezoidal (Crank-Nicolson) method,
- the Runge-Kutta method of order 4.

The implicit methods will require you to solve a nonlinear system of equations.

For each of these methods, consider the following tasks:

a) Using a time step $\Delta t = 500$ seconds, compute a few orbits of the space station and plot the resulting trajectory. What do you observe? Are the orbits closed? What happens if you make the time step smaller?

b) For each of the four methods, compute the position after 100 orbits (this should be approximately or slightly more than 6 days, i.e., 540,000 seconds). Compute $|\mathbf{x}(T_{100\text{ orbits}}) - \mathbf{x}_N)|$ where $\mathbf{x}_N$ is the numerical approximation after a number of time steps that corresponds to an end time of $T_{100\text{ orbits}}$; furthermore, because the exact orbits are closed, the exact solution will satisfy $\mathbf{x}(T_{100\text{ orbits}}) = \mathbf{x}_0$. In other words, we want to know how accurately a method predicts a closed orbit.

   Do this for time step sizes Δt equal to $\frac{T_{100\text{ orbits}}}{10}$, $\frac{T_{100\text{ orbits}}}{100}$, $\frac{T_{100\text{ orbits}}}{1000}$, $\frac{T_{100\text{ orbits}}}{10000}$, and $\frac{T_{100\text{ orbits}}}{100000}$ and determine the convergence rate of each method.

   Note: To answer this question, you will have to think a bit about the exact value of $T_{100\text{ orbits}}$. This value can be computed exactly, or you can get good approximations by just running your most accurate ODE solver for small time steps and extracting $T_{100\text{ orbits}}$ from its results; on the other hand, looking the value up online will not be useful because you will not find it with sufficient accuracy, and it will not exactly match the value that corresponds to your initial conditions. In any case, explain how you chose $T_{100\text{ orbits}}$.

(50 points)