MATH 689: Numerical Optimization

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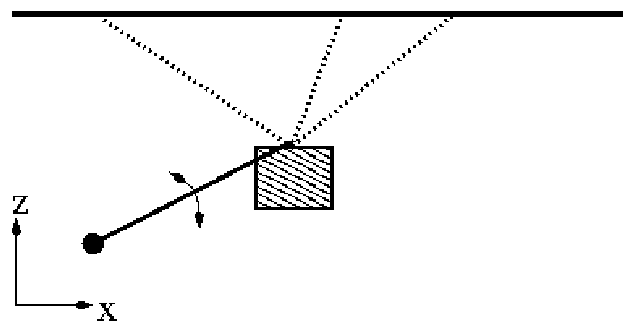
Homework assignment 6 – due Tuesday 3/4/2014

Problem 1 (A modeling exercise, revisited once more). Consider the following system of three springs suspended from the ceiling at positions $(x, z) = (-20\text{cm}, 0\text{cm})$, $(5\text{cm}, 0\text{cm})$ and $(15\text{cm}, 0\text{cm})$ and a rod of fixed length 20cm that is attached at one end at $(-25\text{cm}, -20\text{cm})$ and at the other end to the same point where the springs meet each other:



Each spring has a rest length of $L_0 = 20\text{cm}$, and extending (or compressing) spring $i, i = 1 \dots 3$ to a length L_i requires an energy of $E_{\text{spring},i} = \frac{1}{2} D(L_i - L_0)^2$ where the spring constant for all three springs equals $D = 300 \frac{N}{m}$. On the other hand, the potential energy of the body is $E_{\text{pot}} = mgz$ where the body's mass is $m = 500g$, the gravity constant is $g = 9.81 \frac{\text{m}}{\text{s}^2}$, and z is the vertical coordinate of the body's position.

State this as an optimization problem in standard form (i.e. what is the objective function and what are the constraints). Use the quadratic penalty method to find the solution of the problem that minimizes the energy subject to the constraints. In your answer, show this location as well as the value of the total energy function at this location. Is there only a single energy minimum?

(6 points)

Problem 2 (Lagrange multipliers, part 1). Consider the following minimization problem over $x = (x_1, x_2)^T \in \mathbb{R}^2$:

$$\min_x \ x_2^2$$
$$x_2 - x_1^3 = 0.$$

Answer the following questions: (i) Does it have a solution? (ii) Is the solution unique? (iii) Do the LICQ conditions hold? (iv) Can we derive necessary conditions for the solution that contain a Lagrange multiplier, and if so how do they look? (v) What is the solution to the problem above, and what is the value of the Lagrange multiplier if one exists? (Note: You see me drawing pictures in class all the time to visualize what is going on. Try to understand geometrically how this optimization problem looks like, i.e., in a two-dimensional plot, draw the line where the constraint is satisfied and imagine how the objective function behaves. This will help you understand whether the answers you come up with to the questions above can make sense.) **(4 points)**

Problem 3 (Lagrange multipliers, part 2). In the previous problem, the constraint was that $x_2 = x_1^3$, which traces a cubic curve in $\mathbb{R}^2$ on which the solution must lie. Consider the following variant where instead we take $x_2^2 = x_1^6$ as constraint:

$$\begin{aligned} \min_x \quad & x_2^2 \\ & x_2^2 - x_1^6 = 0. \end{aligned}$$

Describe the constraint geometrically. Answer the same questions as in the previous problem. What changed? **(4 points)**

Problem 4 (Lagrange multipliers, part 3). Consider the following problem

$$\begin{aligned} \min_x \quad & x_2 \\ & x_2^2 + (x_1 - a)^2 - 1 = 0, \\ & x_2^2 + (x_1 + a)^2 - 1 = 0. \end{aligned}$$

For what values of a does this problem have no, one, two, or infinitely many feasible points? If there exist feasible points, for what values of a are the LICQ conditions satisfied? If the LICQ conditions are not satisfied but feasible points exist, does the problem still have a minimum, and if so where is it? For all values of a state the global minimum of this problem. **(3 points)**

If you have comments on the way I teach – in particular suggestions how I can do things better, if I should do more or less examples, powerpoint slides vs whiteboard, etc – or on other things you would like to critique, feel free to hand those in with your homework as well. I want to make this as good a class as possible, and all comments are certainly much appreciated!