MATH 689: Numerical Optimization

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Homework assignment 2 – due Tuesday 2/4/2014

Problem 1 (Convergence order). Determine the order of convergence and the asymptotic error constant for the following sequences:

(a) $a_n = 5.0625, 2.25, 1, \frac{4}{9}, \frac{16}{81}$

(b) $b_n = 2.718, 2.175, 1.740, 1.392, 1.113, 0.8907$

(c) $c_n = 0.318, 0.180, 0.0761, 0.021, 3.04 \cdot 10^{-3}, 1.68 \cdot 10^{-4}, 2.17 \cdot 10^{-6}$.

(3 points)

Problem 2 (Steepest descent iteration). For badly conditioned problems, the steepest descent algorithm takes exceedingly long. Let us verify this claim:

Consider a matrix and vector A, b

$$A = \begin{pmatrix} 10 & 0 \\ 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 10 & 0 \end{pmatrix}.$$

and an objective function

$$f(x) = \frac{1}{2} x^T A x - x^T b.$$

The minimum of this function lies at $x^* = (1, 0)$. Generate graphs that show the surface and contours of the function $f(x)$

Next consider the steepest descent iteration. Start from $x_0 = (2, 10)$. Perform 100 iterations, where in each iteration you compute

$$\delta x_k = -\nabla f(x_k) = b - A x_k, \qquad \alpha_k = \frac{\delta x_k^T \delta x_k}{\delta x_k^T A \delta x_k},$$

and then set $x_{k+1} := x_k + \alpha_k \delta x_k$. Plot the iterates $x_k = (x_{k1}, x_{k2})$ in a 2-dimensional plot and connect them by lines to see their convergence.

How many iterations do you need to achieve an accuracy of $\|x_k - x^*\|_2 \leq 10^{-4}$? Repeat the experiment where a_{11} and b_1 both have the values 1, 10, 100, 1000, 10000 (all other elements of A and b unchanged), and starting from $x_0 = (2, a_{11})$. Create a table with the condition number of these matrices and how many iterations it takes to achieve above accuracy. **(8 points)**

Problem 3 (Newton's method). Repeat the previous problem, but instead of using the steepest descent algorithm use Newton's method with

$$\delta x_k = -[\nabla^2 f(x_k)]^{-1} \nabla f(x_k), \qquad \alpha_k = 1.$$

Explain your observations. **(2 points)**

A problem to think about (Slow convergence of Newton's method). While generally consider very fast, there are cases where even Newton's method makes only very slow progress. Examine the problem of finding the minimum of the one-dimensional function $f(x) = x^{30}$, starting at an arbitrary poing x_0. The minimum, of course, lies at $x^* = 0$. Write down the equation for δx_k, given x_k. In the following, assume that we choose a step length of $\alpha_k = 1$ in every iteration.

For the concrete choice $x_0 = 20$, write a little program that finds the minimum using Newton's method. Plot the distance $|x_k - x^*|$ as a function of the iteration number k. How many function and gradient evaluations do you need to achieve an accuracy of $|x_k - x^*| < 10^{-4}$? What is the convergence order you observe? **(No points)**

Another problem to think about (The power of looking at problems differently). Given data points $\{t_i, y_i\}$ there were different ways to fit a line $y(t) = at + b$ through them. Among them were the least sum of squares, the least sum of absolute values, and the least maximal value objective function. In last week's homework, you had seen that the objective function that corresponds to the latter two was non-smooth. On the other hand, on the slides that were shown during the first two classes, you had seen a trick that can reformulate the least-absolute-values problem from a non-smooth unconstrained one into a constrained problem in which both objective function and constraints were linear – i.e. a problem that is much simpler to solve.

Can you find a way in which the least-maximal-value problem that corresponds to the objective function $f(x) = \max_i |y_i - y(t_i)|$ can be reformulated in a similar way, yielding a linear problem with linear inequalities? If so, compare the number of additional variables and the number of inequalities needed to reformulate the maximal difference and sum of differences problems. **(No points)**

If you have comments on the way I teach – in particular suggestions how I can do things better, if I should do more or less examples, powerpoint slides vs whiteboard, etc – or on other things you would like to critique, feel free to hand those in with your homework as well. I want to make this as good a class as possible, and all comments are certainly much appreciated!