MATH 437: Principles of Numerical Analysis

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Homework assignment 8 – due Thursday 10/31/2013

Problem 1 (Polynomial interpolation, again; last week's homework didn't show what I intended to show, see the answer sheets). Compute the polynomial $p_{2N}(x)$ of order $2N$ such that

- $p_{2N}(0) = 1$,
- $p_{2N}(\pm\frac{j}{N}) = 0$ for $j = 1, \ldots, N$.

Plot these polynomials for $N = 2, 4, 6, 8, 12, 20$ in the interval $-1 \leq x \leq 1$. What happens as N becomes larger? **(3 points)**

Problem 2 (L^∞ norm for functions). For vectors, the l^∞ norm equals the magnitude of the largest component of the vector. Similarly, for a function $f(x), a \leq x \leq b$, we define the infinity norm (now written with an upper-case L^∞ to indicate that this is the norm of a function, rather than a vector) as

$$\|f\|_{L^\infty(a,b)} = \max_{a \leq x \leq b} |f(x)|.$$

Consider the functions $p_{2N}(x)$ computed in Problem 1. These functions are made to interpolate data points (x_i, y_i) for which all data points y_i lie in the range $0 \leq y_i \leq 1$. Yet, as you should have seen from the graphs produced for Problem 1, $p_{2N}(x)$ does not respect this range; the interpolating polynomials oscillate wildly between interpolation points.

For the 6 polynomials $p_{2N}(x)$ computed in Problem 1 for $N = 2, 4, 6, 8, 12, 20$, compute $\|p_{2N}\|_{L^\infty(-1,1)}$.

(Note: The maximum of a function f of course satisfies $f' = 0$. For the polynomial p_{2N} of order $2N$, this means that you are looking for a zero of a polynomial of order $2N - 1$. This problem is not solvable in general if $N \geq 3$ – unless the coefficients of the polynomial satisfy some really lucky coincidence – so I will be satisfied if you can come up with any idea, rigorous or not, to find an approximate value of the infinity norm of p_{2N}, as long as you explain how you compute it.) **(3 points)**

Problem 3 (Least-squares approximation). Take the same $2N+1$ points as in Problem 1. For $N = 6, 8, 12, 20$, compute the best least-squares approximating polynomials of order 4, i.e. the polynomial $p_4(x)$ such that

$$\left(\sum_{i=1}^{N} |p_4(x_i) - y_i|^2 \right)^{1/2}$$

is minimal. Plot them for the range $-1 \leq x \leq 1$. Compare to the corresponding polynomials from Problem 1. What is the behavior of the least-squares approximates between the data points (x_i, y_i)? **(5 points)**

Problem 4 (Extrapolation). We have measured the following 10 data points:

x_i	1	2	3	4	5	6	7	8	9	10
y_i	1.51	2.01	2.49	2.98	3.51	4.01	4.49	5.02	5.52	5.98

It seems reasonable to assume a linear relationship between x and y. Compute

- the interpolating polynomial $p_9^{inter}(x)$ for these 10 data points;
- the linear least-squares polynomial $p_1^{ls}(x)$ that best approximates these data points.

Plot both in the interval $-2 \leq x \leq 12$, together with the data points. If we want to extrapolate the measured behavior (i.e., predict the behavior of y outside the range $1 \leq x \leq 10$ within which we have obtained measurements), what can you conclude from the plots? In particular, what are the values $p_9^{inter}(12)$ and $p_1^{ls}(12)$ that the two functions predict for $x = 12$? What value would you expect from the linear model that was clearly the basis on which the data points were obtained? **(5 points)**