MATH 437: Principles of Numerical Analysis

Prof. Wolfgang Bangerth // Blocker 507D // bangerth@math.tamu.edu
TA: Youli Mao // Blocker 505E // youlimao@math.tamu.edu

Homework assignment 5 – due Thursday 10/03/2013

Problem 1 (Norms on infinite dimensional spaces). In finite dimensional spaces, such as $\mathbb{R}^n$, you have seen that all norms are equivalent, i.e., if $\|\cdot\|$ and $\|\cdot\|'$ satisfy all the norm axioms, then there are constants $0 < c \leq C < \infty$ so that

$$c\|x\| \leq \|x\|' \leq C\|x\| \qquad \forall x \neq 0.$$

This does not hold on infinite dimensional spaces. For example, consider the space $C([a, b])$ of all continuous and bounded functions on the closed interval $[a, b]$ with given and fixed $a < b$. In other words, $C([a, b])$ is the *set* of all functions $f(\cdot)$ that are continuous and that are bounded from below and above. Consider the following two norms:

- For a function $f \in C([a, b])$, let $\|\cdot\|$ be defined by

  $$\|f\| = \max_{x\in[a,b]} |f(x)|.$$

  This norm is typically called the *supremum norm.*

- For a function $f \in C([a, b])$, let $\|\cdot\|'$ be defined by

  $$\|f\|' = \int_a^b |f(x)|\, dx.$$

  This norm is called the $L_1([a, b])$ norm. It is well defined also on a larger set of functions (e.g., they need not be continuous, and some unbounded functions are also allowed).

Answer the following questions:

- Show that both of these are in fact norms, i.e., that they satisfy the norm axioms. **(6 points)**

- Show that these norms are not equivalent, i.e., that there is either no $c > 0$ or no $C < \infty$ (or neither such constant) that satisfies the above inequalities *for all elements of the space* (i.e., here: for all functions).

  Hint: The left inequality, for example, means that there is no $c > 0$ so that $\frac{\|f\|'}{\|f\|} \geq c > 0$ for all nonzero $f \in C([a, b])$. One may be tempted to show this by trying to find a function $f \neq 0$ so that this ratio is zero; this can't work, however, because it would imply $\|f\| = 0$ for a nonzero f, which would mean that $\|\cdot\|$ is not a norm. Rather, one tries to find a sequence $\{f_n\} \subset C([a, b])$ so that $\frac{\|f_n\|'}{\|f_n\|} \to 0$. Given this hint, find a sequence $\{f_n\}$

of continuous and bounded functions for which either the ratio $\frac{\|f_n\|'}{\|f_n\|}$ or $\frac{\|f_n\|}{\|f_n\|'}$ are not bounded away from zero. (Think graphically what the two norms represent.) **(6 points)**

- For finite a, b, you will only be able to find sequences $\{f_n\}$ that violate one of the two inequalities. Is this also true for the space $C(\mathbb{R})$, i.e., for $a = -\infty, b = +\infty$? **(3 points)**

Problem 2 (Jacobi iteration) The linear system

$$\begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

does not fall into the category of linear systems for which we have shown in class that Jacobi's method converges because the matrix is not strictly diagonally dominant. However, we can *modify* the method so that it converges. Consider the *damped* Jacobi iteration:

$$x^{k+1} = x^k + \omega B(b - Ax^k)$$

where $B = [\text{diag}(A)]^{-1}$ and $0 < \omega \leq 1$.

- Show experimentally that Jacobi's method does not converge for $\omega = 1$ but that the damped method does converge for $\omega = 0.5$. Find (experimentally) the range of omegas for which it converges. **(4 points)**

- Prove that the damped method converges if ω is sufficiently small. For the given matrix, can you show how small ω needs to be for the method to converge? **(3 points)**

Bonus problem (Convergence of Jacobi iteration). In class, we have shown that Jacobi's iteration converges to the solution of the linear system $aX = B$ if the matrix A is *strictly* diagonally dominant. The theorem did not say anything about other matrices.

Show that the Jacobi iteration also converges if the matrix is *irreducibly* diagonally dominant. A matrix is irreducibly diagonally dominant if

- for all rows i there holds
$$A_{ii} \geq \sum_{j \neq i} |A_{ij}|$$

- for at least one row the inequality holds as less-than rather than less-than-or-equal

- it is not possible to permute the rows and columns in a way so that the matrix decomposes into two blocks along the diagonal that are not connected via off-diagonal blocks. **(2 bonus points)**