MATH 417: Numerical Analysis

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Homework assignment 4 – due 2/27/07

Problem 1 (Convergence order for functions). Determine the exponent p in the following statements and explain your choice:

a) $12x^4 - 2x^3 \ln x = \mathcal{O}(x^p) \qquad \text{as } x \to \infty.$

b) $\frac{2}{(x-1)^2} - \ln \ln x + \frac{1}{\ln \ln x} = \mathcal{O}((x-1)^p) \qquad \text{as } x \to 1.$

c) Is there an exponent p for which the statement

$$x \ln x = \mathcal{O}(x^p) \qquad \text{as } x \to \infty$$

is true? If not, is there an exponent for which the following holds:

$$x \ln x = o(x^p) \qquad \text{as } x \to \infty.$$

(4 points)

Problem 2 (Gaussian elimination). Solve (on paper, showing the individual steps) the following system of linear equations using Gaussian elimination:

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

Verify that your result is correct.

(The matrix is the example is the so-called Hilbert matrix, with entries $H_{ij} = \frac{1}{i+j-1}$. It has a number of nasty properties that make it a good testcase for matrix algorithms.) **(5 points)**

(please turn over)

Problem 3 (Gaussian elimination). Using Gaussian elimination, it is simple to solve the following problem

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

One would eliminate the occurrence of x_1 in the second equation by subtracting the first from the second equation, arriving at a diagonal matrix.

Describe what happens if the system instead looked like this:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Does the algorithm still work? If not, propose a remedy. **(2 points)**

Problem 4 (Positive definite matrices). Positive definite matrices are those matrices for which $x^T Ax > 0$ for all vectors $x \neq 0$. These matrices play an important role in many applications of engineering and physics. Let us consider one of their properties.

Any matrix A can be written as $A = A^s + A^a$, where the symmetric part A^s and the skew-symmetric part A^a of a matrix are defined as

$$A^s = \frac{A + A^T}{2}, \qquad A^a = \frac{A - A^T}{2}.$$

Prove that if A is positive definite then A^s is positive definite, and vice versa. **(3 points)**

Problem 5 (LU decomposition). Solve the linear system $Ax = b$ with the Hilbert matrix system we already saw in Problem 2:

$$A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

by applying the following steps with paper and pencil:

1. Compute the LU decomposition of A and write down the elimination steps.
2. Use forward and backward substitution to obtain the solution x.

(5 points)