MATH 417: Numerical Analysis

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Homework assignment 3 – due 2/15/2007

Problem 1 (Secant method). This problem is an example of finding the root of a function f that is only given in form of a procedure, a likely case in applications, instead of as a closed form expression.

In order to define the function $g(x)$, consider the following iteration: set $a_0 = 1$ and compute the values a_i by the following iteration:

$$a_i = a_{i-1} + \frac{x \cos a_{i-1} + x}{10}.$$

Clearly, we can compute a_1 from $a_0 = 1$ for each value of x. Similarly, we can compute a_2 from a_1, and so on. Now, let $g(x)$ be the function whose value equals a_{10} for any given value of x.

a) Write a program function that given a value x returns $g(x) = a_{10}$ by computing the iteration above. Use your program to plot $g(x)$ in the range $-10 \leq x \leq 10$.

b) Assume we want to solve the equation $f(x) = 0$ where $f(x) = g(x) - 2$. State why Newton's method may be ill-suited for this task.

c) Write a program that finds a root of $f(x) = g(x) - 2$ up to 6 digits accuracy using the secant method. Use $x_0 = 0, x_1 = 1$ as starting points. State how many iterations you needed to get the desired accuracy. (Hint: Because the problem – probably – doesn't have an analytic solution, the exact location x^* of the root of $f(x)$ is unknown. So how do you know when your iterate x_k is accurate to six digits, i.e. that $x_k - x^* < 10^{-6}$? In order to test algorithms, one often lets them run for a quite significant number of iterations, for example so that between x_k and x_{k+1} the 10th or 12th digit doesn't change any more. If that is the case, then one can be virtually assured that the first 9 or 11 digits of x_{k+1} are correct. You can then use this as a pretty good approximation of x^* and compare the first few iterates against it, to count how many it takes so that the first 6 digits coincide.)

d) Write a program that solves the same problem using the bisection method instead of the secant method, using $a_0 = 0, b_0 = 1$ as the initial interval. State how many iterations it takes this time to get 6 correct digits. How do the secant and the bisection method compare in terms of function evaluations for this problem? **(8 points)**

Problem 2 (Root finding methods). Compare, in words, the bisection method, Newton's method, and the secant method with respect to the following criteria: reliability of finding a root of a function, speed of convergence, complexity (i.e., a method is better if it needs fewer evaluations of $f(x)$ per iteration, or if it only needs function values rather than derivatives).

(3 points)

Problem 3 (Convergence order for sequences). Determine the order of convergence and the asymptotic error constant for the following sequences:

(a) $a_n = 5.0625, 2.25, 1, \frac{4}{9}, \frac{16}{81}$

(b) $b_n = 2.718, 2.175, 1.740, 1.392, 1.113, 0.8907$

(c) $c_n = 0.318, 0.180, 0.0761, 0.021, 3.04 \cdot 10^{-3}, 1.68 \cdot 10^{-4}, 2.17 \cdot 10^{-6}$.

(3 points)

Problem 4 (Convergence order for functions).

(a) For the functions $f(x) = x^4 + 2x^2 + 1$ and $g(x) = x^4 - 2x^2 - 3x$, determine the exponent p in the asymptotic equation

$$f(x) - g(x) = \mathcal{O}(x^p) \qquad \text{as } x \to \infty.$$

(b) For the functions $f(x) = \sin x$ and $g(x) = x$, determine the exponent p in the asymptotic equation

$$f(x) - g(x) = \mathcal{O}(x^p) \qquad \text{as } x \to 0.$$

(Hint: Since we are interested in the limit of x approaching zero, it may make sense to expand $f(x)$ into a Taylor series around $x_0 = 0$.)

(4 points)