MATH 412: Theory of Partial Differential Equations

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Homework assignment 11 – due Thursday 11/29/2007

Problem 1 (Eigenfunction expansion of the wave equation). This problem is similar to Problem 4 on last week's homework. Consider the wave equation

$$\begin{aligned}
\frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} &= q(x,t), \\
u(0,t) &= 0, \\
u(L,t) &= 0, \\
u(x,0) &= f(x), \\
\frac{\partial}{\partial t} u(x,0) &= g(x).
\end{aligned}$$

As for the heat equation, the solution can be written as

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t) \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}.$$

Again, the coefficients $A_n(t)$ have to satisfy a different ordinary differential equation than the coefficients of the wave equation. Go back to your notes to see how the ODE for $a_n(t)$ was derived for the heat equation and adjust this process to the present equation. Derive and explain which ODE $A_n(t)$ has to satisfy here, and how we can derive initial conditions $A_n(0)$ and $\frac{dA_n}{dt}(0)$. Also explain why we need two initial conditions for A_n.

(3 points)

Problem 2 (Solution of the wave equation). The solution to Problem 1 is that the solution to the wave equation in 1-d, assuming a length $L = 1$, is

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t) \sqrt{2} \sin(n\pi x),$$

and that the coefficients $A_n(t)$ are

$$A_n(t) = \alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t) + \int_0^t q_n(t) \frac{\sin(\omega_n t - \omega_n \tau)}{\omega_n} \, d\tau,$$

where $\omega_n = cn\pi$ and

$$\alpha_n = \int_0^1 f(x)\sqrt{2}\sin(n\pi x)\, dx, \qquad \beta_n = \frac{1}{\omega_n}\int_0^1 g(x)\sqrt{2}\sin(n\pi x)\, dx,$$

$$q_n(t) = \int_0^1 q(x,t)\sqrt{2}\sin(n\pi x)\, dx.$$

(Note that we here always use $\phi_n(x) = \sqrt{2}\sin(n\pi x)$, i.e. with the factor $\sqrt{2}$, because this way we have the orthonormality condition $\int_0^L \phi_n(x)\phi_m(x)dx = \delta_{nm}$.)

Compute the terms $\alpha_n, \beta_n, q_n(t)$ for the following set of initial conditions and source terms:

$$q(x,t) = 0, \qquad f(x) = \begin{cases} x & \text{for } x < \frac{1}{2} \\ 1-x & \text{for } x \geq \frac{1}{2} \end{cases}, \qquad g(x) = 0.$$

This corresponds to a string that is picked exactly in the middle and forms a straight line from the two ends (where it is clamped to zero elevation) to the center point.

With these coefficients, state the complete form of the solution to the problem. (Hint: The solution is $\alpha_n = 0$ for even n, $\alpha_n = \frac{4\sqrt{2}}{n^2\pi^2}(-1)^{\frac{n-1}{2}}$ for odd n, $\beta_n = q_n(t) = 0$. However, you should explain why.) **(5 points)**

Problem 3 (A numerical demonstration of the wave equation). In Problem 2, you have derived the solution of the wave equation for a particular set of initial and source conditions. Let the wave speed $c = 1$. Then we had that

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t)\sqrt{2}\sin(n\pi x),$$

where A_n satisfies the equations given in the previous problem. Let us approximate this solution by not taking infinitely many terms, but truncating the sum at a finite number, let's say 20:

$$\tilde{u}(x,t) = \sum_{n=1}^{20} A_n(t)\sqrt{2}\sin(n\pi x).$$

For given values of x and t, you now have all the information to compute $\tilde{u}(x,t)$. Write a computer program that does exactly this. Generate plots of $\tilde{u}(x,t)$ at $t = 0, \frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \dots, \frac{30}{10}$, for example by sampling x in intervals of 0.01. **(4 points)**

Bonus question: 3 extra points for everyone who manages to produce a movie out of this function $\tilde{u}(x,t)$ showing how this string vibrates. 1 further extra point if you take not only the first 20, but the first 100 terms of the infinite sum into account. **(up to 4 points)**

Problem 4 (How long a turkey must roast). Company *Au Naturale* is rolling out their new product just in time: the "□ turkey 2007", a cube-shaped turkey! They're easy to raise at high density, easy to transport to the store, and fit snugly into the oven. The family-size product has dimensions $L = W = H = 20cm$, weighs 8kg and is therefore perfectly suited for a family of mom, pop, kids, and grandparents with plenty left over for the big game on Friday.

The only question *Au Naturale* still needs to answer is how long the instructions should say the turkey needs to be in the oven. To be done, the center needs to reach 165°F. Let's help them figure it out!

To model this situation, remember that the turkey is box shaped with domain $\Omega = [0,L] \times [0,W] \times [0,H] = [0,20]^3 \subset \mathbb{R}^3$. Heating a turkey is described by the following equations:

$$\begin{aligned} \frac{\partial u(x,y,z,t)}{\partial t} - k\Delta u(x,y,z,t) &= 0 && \text{in } \Omega \times [0,T], \\ u(x,y,z,0) &= T_0 && \text{in } \Omega, \\ u(x,y,z,t) &= T_1 && \text{on } \partial\Omega. \end{aligned}$$

Here, $u(x, y, z, t)$ is the temperature of the turkey at location (x, y, z) at time t, with initial temperature T_0 (assumed constant), and boundary temperature T_1 (also assumed constant). $k = \frac{K}{\rho c}$ is the ratio of heat conductivity to density times specific heat.

The easiest way to solve this problem is to introduce a function $d(x, y, z, t) = u(x, y, z, t) - T_1$, i.e. the difference between the temperature at a point (x, y, z) at time t and the oven temperature T_1. Show that d satisfies

$$\begin{aligned} \frac{\partial d(x,y,z,t)}{\partial t} - k\Delta d(x,y,z,t) &= 0 && \text{in } \Omega \times [0,T], \\ d(x,y,z,0) &= T_0 - T_1 && \text{in } \Omega, \\ d(x,y,z,t) &= 0 && \text{on } \partial\Omega. \end{aligned}$$

We've shown in class how to derive the solution for this problem for a general domain, where we use a superposition of modes. Here, the modes are $\phi_n(\mathbf{x})$, where $n = (i, j, k)$ is a multi-index and ϕ_n are the eigenfunctions of the Laplace operator on Ω. For the current domain, these are functions you calculated on a previous homework. Using these functions, state the general solution of the problem in the form

$$d(x,y,z,t) = \sum_n a_n(t)\phi_n(x,y,z),$$

and calculate explicit formulas for the coefficients $a_n(t)$, taking into account the boundary and initial conditions given above.

Next, use your solutions $a_n(t)$ to evaluate the temperature $u(x, y, z, t) = d(x, y, z, t)+T_1$ for the following values of the constants:

- The turkey was in the fridge before being put into the oven, so the initial temperature $T_1 = 278$ K (≈ 40 F);
- The oven is set to $T_1 = 465$ K (≈ 350 F); and
- Realistic values for the coefficients: $\rho = 1\frac{g}{cm^3}, K = 0.5\frac{J}{smK}, c = 4\frac{J}{gK}$.

Finally, evaluate the temperature at the center of the turkey over time, i.e. state explicitly

$$u\left(\frac{L}{2}, \frac{W}{2}, \frac{H}{2}, t\right) = \sum_n a_n(t)\phi_n\left(\frac{L}{2}, \frac{W}{2}, \frac{H}{2}\right) + T_1.$$

Generate a plot of this function and determine the time at which the center temperature exceeds 360 K (≈ 165°F). This is the time the turkey needs to come out of the oven! **(6 bonus points)**

Happy Thanksgiving!