MATH 412: Theory of Partial Differential Equations

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Homework assignment 10 – due Thursday 11/15/2007

Problem 1 (Eigenvalue distributions). For a string of length π, the eigenvalues of the Laplacian are $\lambda_n = n^2$ for $n = 1, 2, \ldots$. For a square of length and height π, they are $\lambda_{k,l} = k^2 + l^2$ for $k = 1, 2, \ldots$ and $l = 1, 2, \ldots$. Finally, in three dimensions, for a box of dimensions π, the eigenvalues are $\lambda_{k,l,m} = k^2 + l^2 + m^2$.

For each of these three cases, do the following:

- Show the values of the 10 smallest eigenvalues.
- Write a program that calculates all eigenvalues that are smaller than 10,000. Let the program count how many of those are in each of the intervals $0 \ldots 99$, $100 \ldots 199$, $200 \ldots 299$, etc. until $9900 \ldots 9999$. Generate a plot of the number of eigenvalues in each of these bins.

(4 points)

Problem 2 (Eigenfunction expansion). Solve problem 8.3.2 in the book, i.e. derive a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{L},$$

with time-dependent coefficients $a_n(t)$. State the differential equations the $a_n(t)$ have to satisfy and derive that the solution must converge to a steady state under the conditions stated in the problem. **(4 points)**

Problem 3 (Eigenfunction expansion). Use the method of eigenfunction expansions to solve the following problem:

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} - k \frac{\partial^2 u(x,t)}{\partial x^2} &= 1, \\ u(0,t) &= 0, \\ u(1,t) &= 0, \\ u(x,0) &= 0. \end{aligned}$$

You may use the formulas from Problem 2, but this time need to solve for the explicit form of the coefficients $a_n(t)$. **(4 points)**

Problem 4 (Eigenfunction expansion of a different equation). Consider the following variant of the heat equation (note the additional term in the PDE):

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} - k\frac{\partial^2 u(x,t)}{\partial x^2} + \alpha u(x,t) &= q(x), \\ u(0,t) &= 0, \\ u(1,t) &= 0, \\ u(x,0) &= f(x).\end{aligned}$$

As for the heat equation, the solution can be written as

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{L}.$$

However, the coefficients $a_n(t)$ now have to satisfy a different ordinary differential equation. Go back to your notes to see how the ODE for $a_n(t)$ was derived for the heat equation and adjust this process to the present equation. State which ODE $a_n(t)$ has to satisfy here. **(3 points)**