MATH 412: Theory of Partial Differential Equations

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Homework assignment 6 – due Thursday 10/11/2007

Problem 1 (Maximum principle). The maximum principle states that the solution of the Laplace equation

$$-\Delta u = 0 \qquad \text{in } \Omega, \qquad\qquad u = g \qquad \text{on } \partial\Omega$$

attains its maximum on the boundary. Describe how this can be used to prove that any solution of the Poisson equation

$$-\Delta v = f \qquad \text{in } \Omega, \qquad\qquad v = h \qquad \text{on } \partial\Omega$$

for given functions f, h is unique. **(3 points)**

Problem 2 (A maximum principle for the heat equation?). Do you think that solutions to the heat equation

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} - \Delta u(x,t) &= 0 && \text{in } \Omega \times [0,T], \\ u(x,t) &= g(x,t) && \text{on } \partial\Omega \times [0,T] \\ u(x,0) &= u_0(x) && \text{in } \Omega \end{aligned}$$

also satisfy a maximum principle of the form that at every fixed time t the solution $u(x,t)$ attains its maximum on the boundary? Why or why not? (To answer this question, think first about what the maximum principle for the Laplace equation meant in physical terms, for example for the equilibrium temperature distribution in an object! The heat equation with zero right hand side describes the time dependent temperature distribution in such an object.)

If there is no such maximum principle, then we can't use this route to prove uniqueness for solutions of the heat equation using this approach. What other approach could we use instead to prove uniqueness? **(3 points)**

Problem 3 (Fourier series). The Fourier series on $[-L, L]$ of a function $f(x)$ that is piecewise smooth is given by

$$A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

where

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx,$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \qquad B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.$$

Calculate the Fourier series on $[-\pi,\pi]$ of the function

$$f(x) = \begin{cases} 1 & \text{for } x \geq 0, \\ -1 & \text{for } x < 0. \end{cases}$$

(5 points)

Problem 4 (Gibbs phenomenon). Consider the same function $f(x)$ and its Fourier series as in the previous problem. Using a computer graphing program such as Maple, Matlab, or Mathematica (or whatever else you deem fit for the task), graph for the range $x \in [-\pi,\pi]$ the following:

- $f(x)$ and the first 3 terms of its Fourier series;
- $f(x)$ and the first 6 terms of its Fourier series;
- $f(x)$ and the first 15 terms of its Fourier series;
- $f(x)$ and the first 30 terms of its Fourier series.

(Give us a printout of the plot or plots.) You will see that the plots of the first terms of the Fourier series approximate $f(x)$ increasingly well, but that there are over- and undershoots around the location where $f(x)$ has a jump (i.e. at $x = 0$). These oscillations are called *Gibbs phenomenon*.

Conjecture what the Fourier series converges to for points $x < 0$, $x = 0$, and $x > 0$. **(5 points)**

Problem 5 (Periodic continuation). Repeat the plots you already generated for Problem 2, but this time show $f(x)$ and the first 3, 6, 15, and 30 terms of its Fourier series in the interval $[-3\pi, 3\pi]$. Interpret what you see.

(2 points)