MATH 412: Theory of Partial Differential Equations

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Homework assignment 2 – due Thursday 9/13/2007

Problem 1 (Some basics). Let $u(x,t)$ be a function of space and time. Prove the following identities by going back to the basics of multivariate calculus (definition of derivatives, definition of integrals, etc):

$$a) \quad \frac{d}{dt}\int_0^L u(x,t)\,dx = \int_0^L \frac{\partial u(x,t)}{\partial t}\,dx,$$

$$b) \quad \frac{d}{dx}\int_0^L u(x,t)\,dx = 0,$$

$$c) \quad u(0,t) = u(L,t) \qquad \text{implies} \qquad \int_0^T \int_0^L \frac{\partial u(x,t)}{\partial x}\,dx\,dt = 0,$$

$$d) \quad u(x,t) = x^2t^2 \qquad \text{implies} \qquad \int_0^T \int_0^L \frac{\partial u(x,t)}{\partial x}\,dx\,dt = \frac{L^2T^3}{3},$$

$$e) \quad \int_0^L u(x,T)\,dx = \int_0^L u(x,0)\,dx + \int_0^T \int_0^L \frac{\partial u(x,t)}{\partial t}\,dx\,dt.$$

(5 points)

Problem 2 (Divergence theorem). If a function $u(x,y,z)$ satisfies $-\Delta u = 0$ at every point of a domain $\Omega \subset \mathbb{R}^3$, show that for any bounded subset $\omega \subset \Omega$

$$\int_{\partial\omega} \mathbf{n} \cdot \nabla u\,ds = 0$$

holds, where $\partial\omega$ is the surface (boundary) of ω. **(2 points)**

Problem 3 (Isocontours of solutions). Solve exercise 1.5.14 in the book. (The correct term that should be used in the book is "isotherms", by the way, but this is immaterial to the question.) **(2 points)**

Problem 4 (Equilibrium temperatures). Solve exercise 1.4.1 in the book, parts b and g. **(2 points)**

Problem 5 (Binary materials). Solve exercise 1.4.3 in the book.

(3 points)

Problem 6 (Heat energy). Solve exercise 1.4.10 in the book. Here, assume that the material constants K_0, c, ρ we introduced in class all have value 1.

(2 points)

Problem 6 (Not quite the heat equation). Flea migration works a bit like heat transport: they hop around randomly, and after a while they're everywhere. In much the same way as for the heat equation, derive an equation and boundary conditions for the density of fleas $u(\mathbf{x}, t)$ under the following premises:

- The space where fleas hop around is a two-dimensional area ("the room").
- Fleas hop around randomly; consequently if the density of fleas to the right of a line is n times higher than to the left, n times as many fleas will cross it from the right to the left than the other way around (think of what such reasoning meant for the heat flux in relation to the temperature).
- The room has walls through which fleas neither leave not enter.

In the process of deriving your equations, identify the coefficients analogous to heat conductivity, density, etc in the equation you get. State "physical" units for each quantity that appears in your equation and make sure that the equation is dimensionally correct.

Intuitively, since fleas neither leave nor enter the room, their total number must be constant. Can you derive this from the equations? **(5 points)**