MATH 412: Theory of Partial Differential Equations

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Partial answers for homework assignment 8

Problem 4 (Wave equation). The total kinetic and potential energy of a vibrating string of length L at any given time t is

$$E(t) = \frac{1}{2}\int_0^L \left(\frac{\partial u(x,t)}{\partial t}\right)^2 + c^2 \left(\frac{\partial u(x,t)}{\partial x}\right)^2 \, dx.$$

Show that the energy is conserved, i.e. that $E(t_1) = E(t_2)$ for any two time instants t_1, t_2 if $u(x,t)$ satisfies the homogenous wave equation

$$\begin{aligned} \frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} &= 0, && \text{in } \Omega \times [0,T], \\ u(0,t) &= 0 && \text{for } t \in [0,T], \\ u(L,t) &= 0 && \text{for } t \in [0,T]. \end{aligned}$$

Hint: First note that $E(t_2) - E(t_1) = \int_{t_1}^{t_2} \frac{\partial}{\partial t} E(t)\, dt$. Then derive what form $\frac{\partial}{\partial t}E(t)$ has by direct differentiation under the integral in the definition of $E(t)$. Then integrate by parts in space and time as necessary and see if you can cancel terms using the wave equation and its boundary values as stated above.

Answer. Let us consider $\frac{d}{dt}E(t)$:

$$\begin{aligned} \frac{d}{dt}E(t) &= \frac{1}{2}\int_0^L \frac{\partial}{\partial t}\left(\frac{\partial u(x,t)}{\partial t}\right)^2 + c^2 \frac{\partial}{\partial t}\left(\frac{\partial u(x,t)}{\partial x}\right)^2 \, dx \\ &= \frac{1}{2}\int_0^L 2\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + 2c^2 \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}\, dx \end{aligned}$$

Integration by parts in the second term yields

$$\begin{aligned} \frac{d}{dt}E(t) &= \int_0^L \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial t}\, dx + c^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\bigg|_{x=0}^{L} \\ &= \int_0^L \frac{\partial u}{\partial t}\left(\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2}\right) dx + c^2 \frac{\partial u(L,t)}{\partial x}\frac{\partial u(L,t)}{\partial t} - c^2 \frac{\partial u(0,t)}{\partial x}\frac{\partial u(0,t)}{\partial t}. \end{aligned}$$

Because $u(x,t)$ satisfies the wave equation, the term in parentheses is zero. Furthermore, because $u(0,t) = u(L,t) = 0$ (i.e. we keep the ends of the string fixed), the velocities at the end points also satisfy $\frac{\partial u(0,t)}{\partial t} = \frac{\partial u(L,t)}{\partial t} = 0$, and the other two terms are then zero as well. We conclude that $\frac{d}{dt}E(t) = 0$.

By integrating the last identity in time from t_1 to t_2, we find that $E(t_1) = E(t_2)$, as claimed.

Problem 5 (Wave equation). Go back in your notes and check on how we showed uniqueness for the heat equation. There, we used the "energy method", where we multiplied the PDE that has to hold for the difference $\delta(x,t) = u_1 - u_2$ of any two solutions, by δ itself, then integrated over time and space, and finally integrated by parts the spatial component.

Attempt to show uniqueness of solutions of the wave equation in 1d,

$$
\begin{aligned}
\frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} &= Q(x,t), && \text{in } \Omega \times [0,T], \\
u(0,t) &= 0 && \text{for } t \in [0,T], \\
u(L,t) &= 0 && \text{for } t \in [0,T], \\
u(x,0) &= f(x) && \text{in } \Omega, \\
\frac{\partial}{\partial t} u(x,0) &= g(x) && \text{in } \Omega.
\end{aligned}
$$

Follow the same steps as before: First, derive the equation that $\delta(x,t)$ has to satisfy. In contrast to the heat equation, next multiply by $\frac{\partial}{\partial t}\delta(x,t)$ instead of by δ, integrate over space and time, and integrate by parts with respect to space. See if you can derive uniqueness of solutions from this.

Answer. Assume there were two solutions u_1, u_2. Then, using the same arguments as always, the difference $\delta(x,t) = u_1(x,t) - u_2(x,t)$ has to satisfy the following equation, boundary conditions and initial conditions:

$$
\begin{aligned}
\frac{\partial^2 \delta(x,t)}{\partial t^2} - c^2 \frac{\partial^2 \delta(x,t)}{\partial x^2} &= 0, && \text{in } \Omega \times [0,T], \\
\delta(0,t) &= 0 && \text{for } t \in [0,T], \\
\delta(L,t) &= 0 && \text{for } t \in [0,T], \\
\delta(x,0) &= 0 && \text{in } \Omega, \\
\frac{\partial}{\partial t} \delta(x,0) &= 0 && \text{in } \Omega.
\end{aligned}
$$

Note how all right hand side are zero. We will now essentially unroll the argument of Problem 4 again, just the other way around. First multiply the PDE by $\frac{\partial \delta}{\partial t}$ and integrate over Ω, to obtain

$$
0 = \int_\Omega \frac{\partial \delta}{\partial t} \left(\frac{\partial^2 \delta}{\partial t^2} - c^2 \frac{\partial^2 \delta}{\partial x^2} \right) dx = \int_\Omega \frac{\partial \delta}{\partial t} \frac{\partial^2 \delta}{\partial t^2} - c^2 \frac{\partial \delta}{\partial t} \frac{\partial^2 \delta}{\partial x^2} \, dx
$$

Now integrate the second term by parts to get

$$0 = \int_\Omega \frac{\partial \delta}{\partial t}\frac{\partial^2 \delta}{\partial t^2} + c^2 \frac{\partial^2 \delta}{\partial t \partial x}\frac{\partial \delta}{\partial x}\; dx - \left. \frac{\partial \delta}{\partial x}\frac{\partial \delta}{\partial t}\right|_{x=0}^{L}.$$

As in Problem 4, we note that $\delta(0,t) = \delta(L,t) = 0$, and therefore also $\frac{\partial}{\partial t}\delta(0,t) = \frac{\partial}{\partial t}\delta(L,t) = 0$. The evaluations at $x = 0$ and $x = L$ are therefore zero.

Furthermore, note that

$$\frac{\partial \delta}{\partial t}\frac{\partial^2 \delta}{\partial t^2} = \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial \delta}{\partial t}\right)^2, \qquad \frac{\partial \delta}{\partial x}\frac{\partial^2 \delta}{\partial x \partial t} = \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial \delta}{\partial x}\right)^2.$$

Consequently,

$$0 = \frac{1}{2}\int_\Omega \frac{\partial}{\partial t}\left(\frac{\partial \delta}{\partial t}\right)^2 + c^2 \frac{\partial}{\partial t}\left(\frac{\partial \delta}{\partial x}\right)^2\; dx = \frac{1}{2}\frac{d}{dt}\int_\Omega \left(\frac{\partial \delta}{\partial t}\right)^2 + c^2 \left(\frac{\partial \delta}{\partial x}\right)^2\; dx.$$

If we integrate the left and right hand sides in time from 0 to an arbitrary time T, we find that

$$\begin{aligned} 0 = \int_0^t 0 \; dt &= \frac{1}{2}\int_0^t \frac{d}{dt}\int_\Omega \left(\frac{\partial \delta}{\partial t}\right)^2 + c^2 \left(\frac{\partial \delta}{\partial x}\right)^2\; dx\; dt \\ &= \frac{1}{2}\left[\int_\Omega \left(\frac{\partial \delta}{\partial t}\right)^2 + c^2 \left(\frac{\partial \delta}{\partial x}\right)^2\; dx\right]_{t=0}^{T}. \end{aligned}$$

In other words,

$$\int_\Omega \left(\frac{\partial \delta(x,T)}{\partial t}\right)^2 + c^2 \left(\frac{\partial \delta(x,T)}{\partial x}\right)^2\; dx = \int_\Omega \left(\frac{\partial \delta(x,0)}{\partial t}\right)^2 + c^2 \left(\frac{\partial \delta(x,0)}{\partial x}\right)^2\; dx.$$

However, because $\delta(x,0) = 0$, the right hand side is zero and we find

$$\int_\Omega \left(\frac{\partial \delta(x,T)}{\partial t}\right)^2 + c^2 \left(\frac{\partial \delta(x,T)}{\partial x}\right)^2\; dx = 0.$$

We have seen something like this several times before: the integral over non-negative squares of something is zero. We have concluded then and conclude now that this is only possible if each of the terms is zero itself, i.e. in particular that

$$\frac{\partial \delta(x,T)}{\partial x} = 0.$$

However, since $\delta(0,T) = \delta(L,T) = 0$, we see that $\delta(x,T) = 0$. Because we have chosen T arbitrarily, this must hold for all times, i.e. $\delta(x,t) = 0$. From this we conclude that necessarily $u_1(x,t) = u_2(x,t)$, i.e. solutions are unique.