MATH 417: Numerical Analysis

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Homework assignment 6 – due 10/12/06 and 10/16/06

Problem 1 (Vector and matrix norms). Solve problems 7.1/1 and 7.1/4.
(4 points)

Problem 2 (Equivalence of norms on $\mathbb{R}^n$). In class, we proved the equivalence of the norms $\|.\|_\infty$ and $\|.\|_2$. Here now, prove the same for $\|.\|_\infty$ and $\|.\|_1$, where

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|.$$

a) Prove that there are indeed constants c, C such that

$$c\|v\|_\infty \le \|v\|_1 \le C\|v\|_\infty.$$

where

$$\|v\|_1 = \sum_i |v_i|,$$
$$\|v\|_\infty = \max_i |v_i|,$$

and where v is an n-dimensional vector in $\mathbb{R}^n$.

b) For vectors v_1, v_2 with $\|v_1\|_1 \le \|v_2\|_1$, does the result of part a) imply that $\|v_1\|_\infty \le \|v_2\|_\infty$? If not, give an example of vectors for which this does not follow. **(4 points)**

Problem 3 (Jacobi iteration). Let A, b be the 100×100 matrix and 100-dimensional vector defined by

$$A_{ij} = \begin{cases} 2.01 & \text{if } i = j, \\ -1 & \text{if } i = j \pm 1, \\ 0 & \text{otherwise}, \end{cases} \qquad b_i = \frac{1}{100} \sin\left(\frac{2\pi i}{50}\right).$$

Apply Jacobi's method to solving $Ax = b$. Write a program that implements the Jacobi method and start with a vector x_0 with randomly chosen elements in the range $0 \leq (x_0)_i \leq 1$ (i.e. with elements generated from what the `rand()` function or a similar replacement returns).

(Hint: It is not necessary to actually store the complete matrix just to multiply with it. Rather, use that the i-th component of the vector Ay is $(Ay)_i = \sum_{j=1}^{n} A_{ij} y_j = 2.01 y_i - y_{i-1} - y_{i+1}$ at least for $2 \leq i \leq n-1$, and obvious modifications for $j = 1$ and $j = n$.)

Run 200 Jacobi iterations and plot the values of $(x^{(k)})_i$ against i for every few iterations, for example $k = 0, 2, 5, 10, 20, 50, 100, 200$. What do you observe?

(5 points)

Problem 4 (Alternative vector norms). Let A be a symmetric and positive definite $n \times n$ matrix. Show that

$$\|x\|_A = \sqrt{x^T A x}$$

is a norm for vectors $x \in \mathbb{R}^n$. (Hint: Use the eigenvalue and eigenvector decomposition of symmetric positive definite matrices.)

(3 points)