MATH 417: Numerical Analysis

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Homework assignment 5 – due 10/5/06 and 10/9/06

Problem 1 (Positive definite matrices). Positive definite matrices are those matrices for which $x^T Ax > 0$ for all vectors x. These matrices play an important role in many applications of engineering and physics. Let us consider one of their properties.

Any matrix A can be written as $A = A^s + A^a$, where the symmetric part A^s and the skew-symmetric part A^a of a matrix are defined as

$$A^s = \frac{A + A^T}{2}, \qquad A^a = \frac{A - A^T}{2}.$$

Show that A being positive definite is equivalent to A^s being positive definite.

(3 points)

Problem 2 (LU decomposition). Solve the linear system $Ax = b$ with the Hilbert matrix system we already saw last week:

$$A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

by applying the following steps with paper and pencil:

(a) Compute the LU decomposition of A and write down the elimination steps.

(b) Use forward and backward substitution to obtain the solution x.

(4 points)

(please turn over)

Problem 3 (LU decomposition). Write a program that implements the LU decomposition algorithm for general $n \times n$ matrices and outputs the L and U factors. Apply it to the matrix of Problem 2.

In a second step, implement the backward and forward substitution solves with the upper and lower triangular factors L and U for any given vector. Apply it to the given right hand side of Problem 2. **(6 points)**

Problem 4 (Norms on $\mathbb{R}^n$). In the analysis of iterative solution methods for linear systems, we will come across different vector norms. A functional $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is called a *norm* if it satisfies the following three conditions:

(a) $\|x\| \geq 0$ for all vectors $x \in \mathbb{R}^n$ and $\|x\| = 0$ if and only if $x = 0$ (positive definiteness);

(b) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$ and all vectors $x \in \mathbb{R}^n$ (scalability);

(c) $\|x + y\| \leq \|x\| + \|y\|$ for all vectors $x, y \in \mathbb{R}^n$ (triangle inequality).

Determine which of the following are norms on $\mathbb{R}^n$ by proving or disproving that they satisfy the three conditions above:

a) $\max_{1\leq 1\leq n} |x_i|$

b) $\max_{2\leq 1\leq n} |x_i|$

c) $\sum_{i=1}^n |x_i|^3$

d) $\left(\sum_{i=1}^n |x_i|^{1/2}\right)^2$

e) $\max\{|x_1 - x_2|, |x_1 + x_2|, |x_3|, |x_4|, \ldots, |x_n|\}$

f) $\sum_{i=1}^n 2^{-i}|x_i|$ **(6 points)**