MATH 412: Theory of Partial Differential Equations

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Homework assignment 9 – due Thursday 11/9/2006

Problem 1 (Eigenfunctions of the Laplacian). In class, we have explicitly derived that the eigenfunctions of the Laplacian operator on a rectangle of length L and height H are

$$\phi_n(x,y) = \sin\frac{k\pi x}{L}\sin\frac{l\pi y}{H},$$

where n was a *multi-index* $n=(k,l)$ of two numbers. In the one-dimensional situation, we had the very convenient formula

$$\frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \delta_{nm}.$$

Noting that the sines here are the eigenfunctions of the Laplacian in 1-d, one could conjecture that the same holds also for 2-d:

$$\frac{2}{L}\frac{2}{H}\int_0^H\int_0^L \phi_n(x,y)\phi_m(x,y)\,dx\,dy = \delta_{nm}.$$

Prove this formula. Note that n,m are both multi-indices, i.e. for example $n=(k,l), m=(p,q)$ and that δ_{nm} is one only if $n=m$, i.e. if $k=p, l=q$. In other words, $\delta_{nm}=\delta_{kp}\delta_{lq}$. **(4 points)**

Problem 2 (Solutions of the 2d wave equation). Using the eigenfunctions of the Laplacian on the unit square $\Omega=[0,1]^2$, and the orthogonality formula of Problem 1, compute the solution to the wave equation of the wave equation

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} - c^2\Delta u(x,y,t) = 0, \qquad \text{in } \Omega\times[0,T],$$
$$u(x,y,t) = 0 \qquad \text{for } x\in\partial\Omega,$$

if we impose initial values

$$u(x,y,0) = 0,$$
$$\frac{\partial u}{\partial t}(x,y,0) = \begin{cases} 1 & \text{for } x<\frac{1}{2} \text{ and } y<\frac{1}{2}, \\ 0 & \text{otherwise.}\end{cases}$$

(4 points)

Problem 3 (Eigenvalues of the Laplacian in 3d). In class we derived the eigenfunctions and eigenvectors of the Laplacian on the two-dimensional rectangle of length L and height H. Use the same techniques to derive the eigenfunctions and eigenvalues of the Laplacian in the three-dimensional box of length L, height H, and width W. **(4 points)**

Problem 4 (Eigenvalue distributions). For a string of length π, the eigenvalues of the Laplacian are $\lambda_n = n^2$ for $n = 1, 2, \ldots$. For a square of length and height π, they are $\lambda_{k,l} = k^2 + l^2$ for $k = 1, 2, \ldots$ and $l = 1, 2, \ldots$. Finally, in three dimensions, for a box of dimensions π, the eigenvalues are $\lambda_{k,l,m} = k^2 + l^2 + m^2$.

For each of these three cases, do the following:

- Show the values of the 10 smallest eigenvalues.
- Write a program that calculates all eigenvalues that are smaller than 10,000. Let the program count how many of those are in each of the intervals $0 \ldots 99$, $100 \ldots 199$, $200 \ldots 299$, etc. until $9900 \ldots 9999$. Generate a plot of the number of eigenvalues in each of these bins.

(4 points)