MATH 412: Theory of Partial Differential Equations

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Homework assignment 5 – due Thursday 10/5/2006

Problem 1 (Solutions of the Laplace equation on an annulus). Using similar techniques as used for the case of a circle, solve the Laplace equation in polar coordinates, i.e.

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}u(r,\theta)\right)-\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}u(r,\theta)=0$$

in the annulus $a < r < b, -\pi \leq \theta < \pi$. Derive the general solution of this equation, disregarding any boundary conditions.

The general solution will contain constants A_n, B_n, C_n, D_n (just like the general solution on the disk contained coefficients A_n, B_n). In a second step, determine these coefficients if we have boundary conditions at the inner and outer edge of the annulus:

$$u(a,\theta) = f(\theta), \qquad u(b,\theta) = g(\theta).$$

Note: This is problem 2.5.8 of the book, for which the appendix lists the solution formulas. You should show how these formulas can be derived using the same techniques as were used when we solved the Laplace equation on the circle in class. **(8 points)**

Problem 2 (Solutions of the Laplace equation on an annulus). Confirm that

$$u(r,\theta) = \frac{\sin\theta}{r}$$

is a solution of the Laplace equation. Produce a computer generated plot of $u(r,\theta)$ on the unit circle (i.e. $0 \leq r < 1, -\pi \leq \theta < \pi$). **(3 points)**

Bonus question: Prove that the lines where $u(r,\theta)$ is constant, i.e. $u(r,\theta) = C$, are circles. Verify this claim first on a contour plot of u. **(2 points)**

(please turn over)

Problem 3 (Maximum principle). The maximum principle states that the solution of the Laplace equation

$$-\Delta u = 0 \qquad \text{in } \Omega, \qquad\qquad u = g \qquad \text{on } \partial\Omega$$

attains its maximum on the boundary. Describe how this can be used to prove that any solution of the Poisson equation

$$-\Delta v = f \qquad \text{in } \Omega, \qquad\qquad v = h \qquad \text{on } \partial\Omega$$

is unique. **(3 points)**

Problem 4 (A maximum principle for the heat equation?). Do you think that solutions to the heat equation

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} - \Delta u(x,t) &= 0 && \text{in } \Omega \times [0,T], \\ u(x,t) &= g(x,t) && \text{on } \partial\Omega \times [0,T] \\ u(x,0) &= u_0(x) && \text{in } \Omega \end{aligned}$$

also satisfy a maximum principle of the form that at every fixed time t the solution $u(x,t)$ attains its maximum on the boundary? Why or why not? (To answer this question, think first about what the maximum principle for the Laplace equation meant in physical terms, for example for the equilibrium temperature distribution in an object! The heat equation with zero right hand side describes the time dependent temperature distribution in such an object.)

If there is no such maximum principle, then we can't use this route to prove uniqueness for solutions of the heat equation using this approach. What other approach could we use instead to prove uniqueness? **(3 points)**