MATH 609-602: Numerical Methods

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Homework assignment 2 – due Tuesday 9/13/2005

Problem 1 (Taylor series). Many important functions such as the sine cannot be computed in a simple way, i.e. with only the four basic operations plus, minus, multiplication and division. However, they can be approximated with these operations.

Graph the first eight Taylor approximations of $f(x) = \sin x$ when expanded around zero, i.e.

$$
\begin{aligned}
f_1(x) &= f(0) + f'(0)x, \\
f_2(x) &= f(0) + f'(0)x + \tfrac{1}{2}f''(0)x^2, \\
f_3(x) &= f(0) + f'(0)x + \tfrac{1}{2}f''(0)x^2 + \tfrac{1}{3!}f'''(0)x^3,
\end{aligned}
$$

etc. What do you observe? What does this mean for the approximation of $f(2\pi)$?

Write a program to experimentally determine the number of terms you need to approximate $f(2\pi) = 0$ to an accuracy of 10^{-4} and 10^{-12}. **(6 points)**

Problem 2 (Computing roots by primitive operations) In Problem 1, you saw how a Taylor series can be used to compute the value of a non-primitive function (i.e. a function that does not only require the four basic operations) can be approximated using only primitive operations.

Consider the non-primitive function $f(x) = \sqrt{x}$. Derive

a) a Taylor series approximation around $x_0 = 1$ to compute $f(x)$.

b) a Newton iteration starting at 1. (Note: Numerically evaluating $f(x)$ for a concrete value of x means finding a number y such that $y = f(x)$. We can find it by finding the root y^* of the function $g(y) = y - f(x)$, using a Newton iteration for y_k starting at y_0 and converging to y^*. In the present case, it may be worthwhile to look for a root of the simpler function $g(y) = y^2 - f(x)^2 = y^2 - x$ instead.)

c) Evaluate $f(2) = \sqrt{2}$ using your answers to parts a and b, by substituting $x = 2$ into your formulas and simplifying terms. **(4 points)**

Problem 3 (Solving nonlinear equations). Assume the height of a satellite over (or under) the horizon in degrees is described by the function

$$f(t) = 58 \sin(2\pi(t - 7)/24) + 17,$$

where $t \in [0, 24]$ is the time in hours after midnight. Write programs to compute the time(s) when the satellite crosses the horizon line, i.e. where $f(t) = 0$. Use:

a) the bisection algorithm

b) Newton's method

c) the secant method

Give at least 8 digits of the result (an accuracy of roughly a millisecond).

How many iterations do each of the three algorithms require to evaluate the sunrise time to 4 correct digits? How many function and gradient evaluations? Plot the accuracy of the iterates x_k against the number of function evaluations.

(8 points)

Problem 4 (Solving nonlinear equations). Assuming that the Riemann Conjecture is true, it is known that the number $\pi(x)$ of primes in the range $1 \ldots x$ can be approximated by

$$\pi(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2}.$$

(See

`http://en.wikipedia.org/wiki/Riemann_hypothesis`

The formula above contains the first two terms of the Taylor expansion of the integral form of $\pi(x)$ given on that webpage.)

Write a program to compute, using Newton's method, the number x for which the approximate formula predicts that there are 10^4 prime numbers less than x, i.e. solve the equation

$$\pi(x) = 10^4.$$

To what accuracy does it make sense to solve this problem? **(4 points)**