

Solutions to the heat equation in some special cases

Consider the heat equation

$$u_t = ku_{xx}$$

with initial condition $u(x, 0) = f(x)$ and the following boundary conditions.

- Zero temperature ends: $u(0, t) = u(L, t) = 0$. The solution is

$$u(x, t) = \sum_{n=1}^{\infty} B_n e^{-(n\pi/L)^2 kt} \sin(n\pi x/L)$$

- Perfectly insulating ends: $u_x(0, t) = u_x(L, t) = 0$. The solution is

$$u(x, t) = \sum_{n=0}^{\infty} A_n e^{-(n\pi/L)^2 kt} \cos(n\pi x/L)$$

- Periodic ends: $u(-L, t) = u(L, t)$ and $u_x(-L, t) = u_x(L, t)$. The solution is

$$u(x, t) = \sum_{n=0}^{\infty} a_n e^{-(n\pi/L)^2 kt} \cos(n\pi x/L) + \sum_{n=1}^{\infty} b_n e^{-(n\pi/L)^2 kt} \sin(n\pi x/L)$$

The coefficients A_n , B_n and a_n , b_n are defined using the initial condition:

- For zero temperature ends, the eigenfunctions are $\sin(n\pi x/L)$, $n \geq 1$, and

$$B_n = \frac{2}{L} \int_0^L f(x) \sin(n\pi x/L) dx.$$

- For perfectly insulating ends, the eigenfunctions are $\cos(n\pi x/L)$, $n \geq 0$, and

$$A_0 = \frac{1}{L} \int_0^L f(x) dx, \quad A_n = \frac{2}{L} \int_0^L f(x) \cos(n\pi x/L) dx \quad (n \geq 1)$$

- For periodic ends, the eigenfunctions are $\sin(n\pi x/L)$, $n \geq 1$, $\cos(n\pi x/L)$, $n \geq 0$, and

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin(n\pi x/L) dx$$

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx, \quad a_n = \frac{1}{L} \int_{-L}^L f(x) \cos(n\pi x/L) dx \quad (n \geq 1)$$

- For zero temperature ends the eigenvalues are $(n\pi/L)^2$, $n \geq 1$
- For perfectly insulating or periodic ends, the eigenvalues are $(n\pi/L)^2$, $n \geq 0$

Note: For periodic ends on $0 \leq x \leq L$, that is, $u(0, t) = u(L, t)$ and $u_x(0, t) = u_x(L, t)$, we get eigenvalues $(2n\pi/L)^2$ and eigenfunctions $\sin(2n\pi x/L)$, $n \geq 1$, $\cos(2n\pi x/L)$, $n \geq 0$, and

$$u(x, t) = \sum_{n=0}^{\infty} c_n e^{-(2n\pi/L)^2 kt} \cos(2n\pi x/L) + \sum_{n=1}^{\infty} d_n e^{-(2n\pi/L)^2 kt} \sin(2n\pi x/L).$$

So to keep the same eigenvalues/eigenfunctions, we changed the interval to $-L \leq x \leq L$.