Math 517 HW 7 Solutions

1. Let $f_n : X \to \mathbb{R}$ be continuous functions converging pointwise to $f : X \to \mathbb{R}$, where (i) X is compact, (ii) f is continuous, and (iii) $\{f_n\}$ is monotone decreasing in n. Then $\{f_n\}$ converges to f uniformly. *Give examples to show this result fails if any one of conditions (i)–(iii) is false.*

Solution. (i) Consider $f_n(x) = x^n$ defined on the non-compact set $X = [0,1)$. These are continuous functions that decrease monotonically in n and converge pointwise to the continuous function $f \equiv 0$. The convergence is not uniform: for any n, $|x^n - 0| \geq 1/2$ when $x \in (\sqrt[n]{1/2}, 1) \subseteq X$. (ii) Consider $f_n(x) = x^n$ on $X = [0,1]$, which are continuous functions defined on a compact set that decrease monotonically in n. Their pointwise limit is $f(x) = 0$ for $x \in [0,1)$ and $f(1) = 1$, which is discontinuous at $x = 1$. The convergence is not uniform by the same argument as in (i). (iii) Consider the non-monotonic sequence $f_n(x) = nx$ for $0 \leq x < 1/n$, $f_n(x) = 2 - nx$ for $1/n \leq x < 2/n$, $f_n(x) = 0$ for $2/n \leq x \leq 1$. These functions are continuous, defined on the compact set $X = [0,1]$, and converge pointwise but not uniformly to $f \equiv 0$ (c.f. in-class midterm 2).

2. Let X be compact and consider $Z = \{\text{continuous functions } X \to \mathbb{R}\}$ with the sup metric $d_Z(f,g) = \sup_{x\in X} |f(x) - g(x)|$. Show that Z is complete but not compact.

Solution. Recall convergence in Z is equivalent to uniform convergence. As $f_n(x) \equiv n$ has no subsequence converging pointwise or uniformly, Z is not compact. Let $\{f_n\}$ be a Cauchy sequence in Z. Let $\epsilon > 0$ and pick N such that $n, m \geq N$ implies $d_Z(f_n, f_m) = \sup_{x\in X} |f_n(x) - f_m(x)| < \epsilon$. Then $n, m \geq N$ implies $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in X$. Thus, $\{f_n(x)\}$ is Cauchy for each $x \in X$. Since $\mathbb{R}$ is complete, $\{f_n\}$ has a pointwise limit f. Consider the same ϵ and N as above. Using pointwise convergence, for each $x \in X$ choose $M_x \geq N$ such that $|f(x) - f_{M_x}(x)| < \epsilon$. Then $|f(x) - f_n(x)| \leq |f(x) - f_{M_x}(x)| + |f_{M_x}(x) - f_n(x)| < 2\epsilon$ whenever $n \geq N$ and $x \in X$. This shows $\{f_n\}$ converges uniformly to f. It follows that f is continuous and so $f \in Z$. Thus Z is complete.

3. Suppose $\phi : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous, and there is $L > 0$ such that $|\phi(t,x) - \phi(t,y)| \leq L|x-y|$ for all $t \in [0,T]$ and $x, y \in \mathbb{R}$. Let Z be as in Problem 2 with $X = [0,\epsilon] \subseteq [0,T]$, and for $f \in Z$ define $\Psi : Z \to Z$ by $\Psi(f) \equiv \Psi_f$ where $\Psi_f(t) = c + \int_0^t \phi(s, f(s))\, ds$ and $c \in \mathbb{R}$ is constant. Conclude that for $\epsilon > 0$ sufficiently small the following differential equation has a unique solution:

$$y'(t) = \phi(t, y(t)), \quad y(0) = c, \quad t \in [0,\epsilon] \qquad (*)$$

Solution. If f is continuous, so is $t \mapsto \phi(t, f(t))$ and so Ψ_f is continuous. Thus $\Psi(Z) \subseteq Z$. Also,

$$\begin{aligned} d_Z(\Psi_f, \Psi_g) &= \sup_{t\in[0,\epsilon]} |\Psi_f(t) - \Psi_g(t)| = \sup_{t\in[0,\epsilon]} \left| \int_0^t [\phi(s,f(s)) - \phi(s,g(s))]\, ds \right| \\ &\leq \sup_{t\in[0,\epsilon]} \int_0^t |\phi(s,f(s)) - \phi(s,g(s))|\, ds \leq \sup_{t\in[0,\epsilon]} L \int_0^t |f(s) - g(s)|\, ds \\ &\leq \epsilon L \sup_{t\in[0,\epsilon]} |f(t) - g(t)| = \epsilon L d_Z(f,g). \end{aligned}$$

By the fundamental theorem of calculus, $f \in Z$ is a solution to $(*)$ iff $f = \Psi(f)$, that is, f is a fixed point of Ψ. When $\epsilon L < 1$, the Banach fixed point theorem shows Ψ has a unique fixed point.

4. Let $\{f_n\}$ be an equicontinuous sequence of real-valued functions defined on a compact set X, and suppose $\{f_n\}$ converges pointwise on X. Prove that $\{f_n\}$ converges uniformly on X.

Solution. Let $\epsilon > 0$. Using equicontinuity, pick $\delta > 0$ so that $|f_n(x) - f_n(y)| < \epsilon/3$ whenever $n \geq 1$ and $x, y \in X$ with $d(x,y) < \delta$. Using pointwise convergence, for each $x \in X$ choose N_x such that $n, m \geq N_x$ implies $|f_n(x) - f_m(x)| < \epsilon/3$. Using compactness of X, extract a finite subcover $\{B_\delta(x_i)\}_{i=1}^j$ from $\{B_\delta(x)\}_{x\in X}$. Let $N = \max_{i=1,\ldots,j} N_{x_i}$. For $y \in X$ and $n, m \geq N$, pick x_i with $y \in B_\delta(x_i)$ to get $|f_n(y) - f_m(y)| \leq |f_n(y) - f_n(x_i)| + |f_n(x_i) - f_m(x_i)| + |f_m(x_i) - f_m(y)| < \epsilon$. This shows $\{f_n\}$ is uniformly Cauchy (*i.e.*, Cauchy in Z), hence uniformly convergent, c.f. Problem 2.