Math 517 HW 4 solutions

1. Let (X,d) be a metric space. Define a metric on $X\times X$ that makes $d: X\times X\to\mathbb{R}$ continuous. Show that the distance between compact sets A and B is attained by a pair of points.

Solution. For $(x,y),(z,w)\in X\times X$, define $d_{X\times X}((x,y),(z,w)) = d(x,z)+d(y,w)$. Note that $d_{X\times X}((x,y),(z,w)) \le d(x,u)+d(u,z)+d(y,v)+d(v,w) = d_{X\times X}((x,y),(u,v))+d_{X\times X}((u,v),(z,w))$ for any $(x,y),(u,v),(z,w)\in X\times X$. Thus $d_{X\times X}$ satisfies the triangle inequality. It is easy to see $d_{X\times X}$ satisfies the other properties of a metric. By the triangle inequality for d,

$$d(x,y)-d(z,w) \le d(x,z)+d(z,y)-(d(z,y)-d(y,w)) = d(x,z)+d(y,w).$$

By symmetry, $|d(x,y)-d(z,w)| \le d(x,z)+d(y,w) = d_{X\times X}((x,y),(z,w))$. Thus d is (Lipschitz) continuous. Let $\{(x_n,y_n)\}$ be a sequence in $A\times B$. By compactness of A,B, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to $x\in A$, and a subsequence $\{y_{m_l}\}$ of $\{y_{n_k}\}$ converging to $y\in B$. Then $d_{X\times X}((x_{m_l},y_{m_l}),(x,y)) = d(x_{m_l},x)+d(y_{m_l},y)\to 0$ as $l\to\infty$. This shows $A\times B$ is compact in $(X\times X, d_{X\times X})$. Thus $S := d(A\times B)$ is compact so that $\inf S\in S$ as desired.

2. Suppose $f:\mathbb{R}\to\mathbb{R}$ has the property that whenever $f(x)<w<f(y)$, there exists z between x,y such that $f(z)=w$. Assume for all $r\in\mathbb{Q}$, $f^{-1}(\{r\})$ is closed. Show f is continuous.

Solution. Suppose f is not continuous at $x\in\mathbb{R}$. Then there is $\epsilon>0$ and $\{x_n\}$ in $\mathbb{R}$ converging to x such that $f(x_n)\notin(f(x)-\epsilon, f(x)+\epsilon)$ for all n; WLOG suppose $f(x_n)\ge f(x)+\epsilon$ for all n. Choose $r\in\mathbb{Q}$ with $f(x_n)>r>f(x)$ for all n, and pick t_n between x,x_n such that $f(t_n)=r$. As x is a limit point of $\{t_n : n\in\mathbb{N}\}\subseteq f^{-1}(\{r\})$ and $f^{-1}(\{r\})$ is closed, $f(x)=r$, contradiction.

3. <u>**Using definitions**</u>, show $f:[0,\infty)\to\mathbb{R}$, $f(x)=\sqrt{x}$, is uniformly but not Lipschitz continuous.

Solution. Note that for any $K>0$, $|\sqrt{x}-\sqrt{0}|\le K|x-0|$ is false for $0\le x<1/K^2$. So f is not Lipschitz continuous. To show uniform continuity let $\epsilon>0$ and take $\delta=\epsilon^2$. Then for $0\le y<x$, $|x-y|<\delta \quad \text{implies} \quad |\sqrt{x}-\sqrt{y}|^2 \le |\sqrt{x}-\sqrt{y}||\sqrt{x}+\sqrt{y}| = |x-y| < \delta = \epsilon^2$.

4. Let $f:X\to Y$ be continuous and X compact. Finish the following argument to prove that f is uniformly continuous: If f is not uniformly continuous, then for some $\epsilon>0$ there exists sequences $\{p_n\}$ and $\{q_n\}$ in X such that $\lim_{n\to\infty} d_X(p_n,q_n)=0$ but $d_Y(f(p_n),f(q_n))\ge\epsilon$ for all n...

Solution. Problem 1 shows there is a subsequence $\{(p_{n_k},q_{n_k})\}$ of $\{(p_n,q_n)\}$ converging to (p,q) in $X\times X$ (endowed with the metric $d_{X\times X}$). Then $\{p_{n_k}\}$ and $\{q_{n_k}\}$ converge to p and q respectively, as $d(p_{n_k},p)+d(q_{n_k},q) = d_{X\times X}((p_{n_k},q_{n_k}),(p,q))\to 0$ as $n\to\infty$. Moreover, $p=q$ since $d_X(p,q)\le d_X(p,p_{n_k})+d_X(p_{n_k},q_{n_k})+d_X(q_{n_k},q)\to 0$ as $k\to\infty$. Continuity of f implies $d_Y(f(p_{n_k}),f(q_{n_k}))\le d_Y(f(p_{n_k}),f(p))+d_Y(f(q),f(q_{n_k}))\to 0$ as $n\to\infty$, contradiction.

5. Let $f:(a,b)\to\mathbb{R}$ be monotone increasing and for $x\in(a,b)$ define $L(x)=\sup\{f(z): a<z<x\}$ and $U(x)=\inf\{f(z): x<z<b\}$. Show that: (i) $L(x)\le U(x)$ for all $x\in(a,b)$; (ii) if $a<x<y<b$ then $U(x)\le L(y)$; (iii) f is not continuous at x if and only if $L(x)<U(x)$.

Solution. Throughout fix x and consider $x,y,z,w\in(a,b)$. (i) Note that $f(x)$ is an upper and lower bound for $\{f(z): z<x\}$ and $\{f(z): x<z\}$, respectively. Thus $L(x)\le f(x)\le U(y)$. (ii) Let $x<y$ and pick $w\in(x,y)$. Then since $f(w)\in\{f(z): x<z\}$ and $f(w)\in\{f(z): z<y\}$, we have $U(x)\le f(w)\le L(y)$. (iii) Using (i)-(ii), $f(z)\le U(z)\le L(x)\le f(x)\le U(x)\le L(w)\le f(w)$ when $a<z<x<w<b$; if f is continuous at x, letting $w,z\to x$ shows $L(x)=U(x)$. Conversely suppose $L(x)=f(x)=U(x)$ and let $\epsilon>0$. Choose $a<z<x$ (resp. $x<w<b$) such that $f(z)>L(x)-\epsilon/2$ (resp. $f(w)<U(x)+\epsilon/2$). Then for $\delta=\min\{x-z,w-x\}$ and $y\in(x-\delta,x+\delta)$,

$$f(x)-\epsilon/2 = L(x)-\epsilon/2 < f(z)\le f(y)\le f(w) < U(x)+\epsilon/2 = f(x)+\epsilon/2.$$