1 Solutions to selected problems

1. Let $A \subset B \subset \mathbb{R}^n$. Show that $\text{int } A \subset \text{int } B$ but in general $\text{bd } A \not\subset \text{bd } B$.

Solution. Let $x \in \text{int } A$. Then there is $\epsilon > 0$ such that $B_\epsilon(x) \subset A \subset B$. This shows $x \in \text{int } B$. If $A = [0, 1]$ and $B = [0, 2]$, then $\text{bd } A = \{0, 1\} \not\subset \text{bd } B = \{0, 2\}$.

2. Let $A \subset \mathbb{R}^n$ be open and $f : A \to \mathbb{R}$ continuous with $f(u) > 0$. Show there is an open ball $B$ around $u$ such that $f(x) > f(u)/2$ for $x \in B$.

Solution. Since $f$ is continuous at $u$, for $\epsilon = f(u)/2$ there is $\delta > 0$ such that $f(B_\delta(u)) \subset (f(u) - \epsilon, f(u) + \epsilon)$. In particular, $f(B_\delta(u)) > f(u)/2$ and we may take $B = B_\delta(u)$.

3. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and $f(u) > 0$ if $u$ has at least one rational component. Prove that $f(u) \geq 0$ for all $u \in \mathbb{R}^n$.

Solution. Let $u \in \mathbb{R}^n$ be arbitrary. Write $u = (u_1, \ldots, u_n)$ and let $\{r_k\}$ be the first $k$ digits in the decimal expansion for $u_1$, and let $u_k = (r_k, u_2, \ldots, u_n)$. Then $\{u_k\} \to u$ so by continuity of $f$, $\{f(u_k)\} \to f(u)$. Also $f(u_k) > 0$ for all $k$ since the $r_k$’s are rational. Thus, $f(u) \geq 0$.

4. Show that an open ball in $\mathbb{R}^n$ is bounded.

Solution. Let $B_r(x) \subset \mathbb{R}^n$ be an arbitrary ball. We must show it is contained in a ball $B_s(0)$ around 0. Let $s = r + \|x\|$. If $y \in B_r(x)$ then $\|y\| \leq \|y - x\| + \|x\| < r + \|x\| = s$. Thus, $B_r(x) \subset B_s(0)$.

5. Let $f : A \to \mathbb{R}$ be continuous with $A \subset \mathbb{R}^n$. If $A$ is bounded, is $f(A)$ bounded? If $A$ is closed, is $f(A)$ closed?

Solution. The answer to both questions is no. Consider $f(x) = 1/x$. Then $A = (0, 1)$ is bounded but $f(A) = (1, \infty)$ is not; $A = [1, \infty)$ is closed but $f(A) = (0, 1]$ is not.

6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous with $f(u) \geq \|u\|$ for every $u \in \mathbb{R}^n$. Prove that $f^{-1}([0, 1])$ is sequentially compact.

Solution. Since $[0, 1]$ is closed and $f$ is continuous, $f^{-1}([0, 1])$ is closed. It remains to
show $f^{-1}([0, 1])$ is also bounded, hence sequentially compact. If $x \in f^{-1}([0, 1])$, then $0 \leq f(x) \leq 1$ and $f(x) \geq \|x\|$, so in particular, $\|x\| \leq 1$. Thus, $f^{-1}([0, 1])$ is bounded.

7. Let $A \subset \mathbb{R}^n$ be sequentially compact and $v \in \mathbb{R}^n \setminus A$. Prove there is $u \in A$ such that

$$\|u - v\| \leq \|x - v\| \text{ for all } x \in A.$$  \hspace{1cm} (1)

**Solution.** Let $d = \inf_{x \in A} \|x - v\|$. Since $d + 1/k$ is not a lower bound for $\|x - v\|$ over $x \in A$, we may pick $u_k \in A$ such that $d \leq \|u_k - v\| < d + 1/k$. Using sequential compactness, pick $\{u_n\} \to u$. By continuity of vector subtraction and the norm $\|\cdot\|$, $\{|u_n - v|\} \to \|u - v\|$. And by the squeeze theorem in $\mathbb{R}$, $\|u - v\| = d$.

The point $u$ is not unique: if $A = \{1, -1\} \subset \mathbb{R}$ and $v = 0$, then $u = \pm 1$ satisfy (1).

8. A mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous if there is $K > 0$ such that

$$|F(x) - F(y)| \leq K\|x - y\|$$  \hspace{1cm} (2)

for all $x, y \in \mathbb{R}^n$. Show that a Lipschitz mapping is uniformly continuous.

**Solution.** Suppose (2) holds for $F$. Let $\epsilon > 0$ and pick $\delta = K/\epsilon$. Then

$$\|F(x) - F(x)\| \leq K\|x - y\| < K\delta = \epsilon \text{ whenever } \|x - y\| < \delta.$$

9. Let $A$ be sequentially compact and $f : A \to f(A)$ continuous and injective. Show $f^{-1}$ is continuous. Give an example to show the assumption on $A$ is necessary.

**Solution.** Let $\{v_k\}$ be a sequence in $f(A)$ such that $\{v_k\} \to v$, and let $u_k = f^{-1}(v_k)$, $u = f^{-1}(v)$. Suppose $\{u_k\} \not\to u$. Using sequential compactness of $A$, pick $\{u_n\} \to w \in A$ with $w \neq u$. By continuity of $f$, $\{f(u_n)\} \to f(w)$. As $\{f(u_n)\} = \{v_n\} \to v$, we have $f(w) = v$. Thus, $w = f^{-1}(v) = u$, contradiction.

To see that sequential compactness of $A$ is needed, let

$$f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 4 - x, & 2 \leq x \leq 3 \end{cases},$$

$$f^{-1}(x) = \begin{cases} x, & 0 \leq x < 1 \\ 3 - x, & 1 \leq x \leq 2 \end{cases}.$$  

Note that $f$ is continuous on $A = [0, 1) \cup [2, 3]$ but $f^{-1}$ is not continuous at 1.
10. Let \( f : A \to \mathbb{R}^m \) be continuous and \( A \) sequentially compact. Show \( f \) is uniformly continuous.

**Solution.** Let \( \{u_k\}, \{v_k\} \) be sequences in \( A \) such that \( \{\|u_k - v_k\|\} \to 0 \). Suppose \( \{\|f(u_k) - f(v_k)\|\} \not\to 0 \). Then along a subsequence, \( \{\|f(u_{n_k}) - f(v_{n_k})\|\} \to c > 0 \). (Why?) Since \( A \) is sequentially compact, we can pick sub-subsequences \( \{u_{m_{n_k}}\} \to u \in A \) and \( \{v_{m_{n_k}}\} \to v \in A \), and \( u = v \) since \( \{\|u_k - v_k\|\} \to 0 \). By continuity of \( f \), \( \{f(u_{m_{n_k}})\} \to f(u) \) and \( \{f(v_{m_{n_k}})\} \to f(v) = f(u) \). Thus, \( \{\|f(u_{m_{n_k}}) - f(v_{m_{n_k}})\|\} \to 0 \), contradiction.

11. We say \( u \in \mathbb{R}^n \) is a limit point of \( A \subset \mathbb{R}^n \) if there is a sequence in \( A \setminus \{u\} \) that converges to \( u \). Prove that \( u \) is a limit point of \( A \) if and only if every open ball around \( u \) contains infinitely many points of \( A \setminus \{u\} \).

**Solution.** Let \( u \) be a limit point of \( A \) and \( \delta > 0 \). Let \( \{u_k\} \in A \setminus \{u\} \) be such that \( \{u_k\} \to u \), and pick \( N \) such that \( k \geq N \) implies \( u_k \in B_\delta(A) \setminus \{u\} \). Notice \( \{u_k : k \geq N\} \) must be an infinite set: if it were finite with elements \( v_1, \ldots, v_n \), then for \( \epsilon = \min_{1 \leq i \leq n} \|v_i - u\| \) we would have \( u_k \notin B_\epsilon(u) \) for every \( k \geq N \).

12. Let \( A \subset \mathbb{R}^n \) and let \( f \) be the characteristic function of \( A \). Show that \( f \) is continuous at \( u \) if and only if \( u \notin \text{bd} \, A \). Can one make an analogous statement about \( \lim_{x \to u} f(x) \)?

**Solution.** Suppose \( u \notin \text{bd} \, A \). Then if \( u \in A \), there is \( \delta > 0 \) such that \( B_\delta(u) \subset A \). Given any \( \epsilon > 0 \), if \( \|x - u\| < \delta \) then \( x \in A \) and so \( |f(x) - f(u)| = |1 - 1| = 0 < \epsilon \). If \( u \notin A \), there is \( \delta > 0 \) such that \( B_\delta(u) \subset \mathbb{R}^n \setminus A \) and an analogous argument holds. Conversely, if \( u \in \text{bd} \, A \), then for each \( k \in \mathbb{N} \) there is \( v_k, w_k \in B_{1/k}(u) \) such that \( v_k \in A \), \( w_k \in \mathbb{R}^n \setminus A \). Then \( \{v_k\} \to u \) and \( \{w_k\} \to u \), but \( \{f(v_k)\} \to 1 \), \( \{f(w_k)\} \to 0 \).

An analogous statement does not hold for limits. It is true that if \( u \notin \text{bd}(A) \) then \( \lim_{x \to u} f(x) \) exists – in the argument above, just replace \( \|x - u\| < \delta \) with \( 0 < \|x - u\| < \delta \). However, the converse is false: if \( A = \{0\} \) then \( 0 \in \text{bd}(A) \) yet \( \lim_{x \to 0} f(x) = 0 \) exists. (In the above argument, \( v_k \) and \( w_k \) could equal \( u \).)

13. Let \( f : A \to \mathbb{R}^m \) with \( u \in A \) a limit point of \( A \subset \mathbb{R}^n \). Show that if \( f \) does not have a limit at \( u \), then \( f \) is not continuous at \( u \).

**Solution.** We prove the contrapositive. Suppose \( f \) is continuous at \( u \). Then \( \lim_{x \to u} f(x) = f(u) \). To see this, let \( \epsilon > 0 \) and pick \( \delta > 0 \) such that \( \|f(x) - f(u)\| < \epsilon \) whenever \( \|x - u\| < \delta \) and \( x \in A \); then in particular, \( \|f(x) - f(u)\| < \epsilon \) whenever \( 0 < \|x - u\| < \delta \) and \( x \in A \).
14. Let \( f : A \to \mathbb{R}^n \) with \( u \in A \) a limit point of \( A \subset \mathbb{R}^n \). Show that \( f \) is continuous at \( u \) if and only if \( \lim_{x \to u} f(x) = f(u) \).

*Solution.* We only need to observe that for any \( \epsilon > 0 \), the statements

\[
\|f(x) - f(u)\| < \epsilon \quad \text{whenever} \quad x \in A \quad \text{and} \quad 0 < \|x - u\| < \delta
\]

and

\[
\|f(x) - f(u)\| < \epsilon \quad \text{whenever} \quad x \in A \quad \text{and} \quad \|x - u\| < \delta
\]

are equivalent, since \( \|f(x) - f(u)\| = 0 < \epsilon \) when \( \|x - u\| = 0 \).

15. Let \( A \subset \mathbb{R}^n \) and suppose 0 is a limit point of \( A \). Suppose \( f : A \to \mathbb{R} \) is such that \( f(x) \geq c\|x\|^2 \) for all \( x \in A \), where \( c > 0 \) is constant. Suppose \( g : A \to \mathbb{R} \) is such that \( \lim_{x \to 0} g(x)/\|x\|^2 = 0 \). Prove there is \( r > 0 \) such that \( f(x) - g(x) \geq (c/2)\|x\|^2 \) for \( x \in A \) with \( 0 < \|x\| < r \).

*Solution.* We use \( \epsilon = c/2 \) in the definition \( \epsilon - \delta \) definition of \( \lim_{x \to 0} g(x)/\|x\|^2 = 0 \). Pick \( \delta > 0 \) so that if \( x \in A \) with \( 0 < \|x\| < \delta \), then \( \|g(x)/\|x\|^2\| = \|g(x)/\|x\|^2 < c/2 \) and so

\[
\|f(x) - g(x)\| \geq \|f(x)\| - \|g(x)\| \geq c\|x\|^2 - (c/2)\|x\|^2 = (c/2)\|x\|^2.
\]

16. Let

\[
f(x, y) = \begin{cases} \frac{xy^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}.
\]

Show \( f \) is continuous at \((0, 0)\) and has directional derivatives at \((0, 0)\) in every direction, but is not differentiable at \((0, 0)\).

*Solution.* To see \( f \) is continuous at \((0, 0)\): for \((x, y) \neq (0, 0)\),

\[
|f(x, y) - f(0, 0)| = \left| \frac{xy^2}{x^2 + y^2} - 0 \right| = \frac{|x|}{x^2 + y^2 + 1} \leq |x| \to 0 \text{ as } (x, y) \to 0.
\]

(See problem 14 above.) For the directional derivatives \( D_{(a,b)} f(0,0) \):

\[
\lim_{t \to 0} \frac{f((0,0) + t(a,b)) - f(0,0)}{t} = \lim_{t \to 0} \frac{f(ta, tb)}{t} = \lim_{t \to 0} \frac{t^2ab^2}{t^2a^2 + t^2b^2} = \frac{ab^2}{a^2 + b^2}.
\]

Now note that \( D_1 f(0,0) = 0, D_2 f(0,0) = 0 \) and

\[
\frac{f((0,0) + (x,y)) - f(0,0) - 0x - 0y}{\| (x,y) \|} = \frac{xy^2}{x^2 + y^2} - \frac{0x - 0y}{\sqrt{x^2 + y^2}} = \frac{xy^2}{(x^2 + y^2)^{3/2}}.
\]
When \( x \equiv y \) and \( y \to 0 \) the limit of (3) is \( 2^{-3/2} \neq 0 \), so \( f \) is not differentiable at \((0, 0)\).

17. Define \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) by

\[
f(x, y) = \begin{cases} 
\frac{xy}{x^2+y^2}, & (x, y) \neq (0, 0) \\
0, & (x, y) = (0, 0)
\end{cases}
\]

Show the partial derivatives of \( f \) are not continuous at \((0, 0)\).

**Solution.** For \((x, y) \neq (0, 0)\), we can calculate partial derivatives “as usual” (i.e., without resorting to the definition):\(^1\)

\[
D_1 f(x, y) = \frac{y^3 - x^2y}{(x^2+y^2)^2}, \quad D_2 f(x, y) = \frac{x^3 - xy^2}{(x^2+y^2)^2}.
\]

In particular, \( D_1 f(0, y) = y^{-1}, D_1 f(x, 0) = x^{-1} \) do not have limits as \( y \to 0, x \to 0 \). Thus, they are not continuous at \((0, 0)\). (See problem 13 above.)

18. Define \( g : \mathbb{R}^2 \rightarrow \mathbb{R} \) by

\[
g(x, y) = \begin{cases} 
x^2y^4, & (x, y) \neq (0, 0) \\
0, & (x, y) = (0, 0)
\end{cases}
\]

Is \( g \) continuously differentiable?

**Solution.** For \((x, y) \neq (0, 0)\) we can calculate partial derivatives “as usual”:

\[
D_1 g(x, y) = \frac{2xy^6}{(x^2+y^2)^2}, \quad D_2 g(x, y) = \frac{4x^4y^3 + 2x^2y^5}{(x^2+y^2)^2}.
\]

Moreover, from the limit definition of partial derivatives,

\[
D_1 g(0, 0) = \lim_{t \to 0} \frac{g(t, 0) - g(0, 0)}{t} = \lim_{t \to 0} \frac{0 - 0}{t} = 0,
\]

\[
D_2 g(0, 0) = \lim_{t \to 0} \frac{g(0, 0) - g(0, t)}{t} = \lim_{t \to 0} \frac{0 - 0}{t} = 0.
\]

Since \( D_1 g(x, y) \) and \( D_2 g(x, y) \) are rational functions, they are continuous except at their asymptotes \((x, y) = (0, 0)\). Thus, to establish continuity of \( D_1 g \) and \( D_2 g \) we need only

\(^1D_i f \equiv D_{e_i} f \equiv \frac{\partial f}{\partial x_i}\)
check that \( \lim_{(x,y) \to (0,0)} D_i g(x, y) = 0 \) for \( i = 1, 2 \). For \( (x, y) \neq (0,0) \),

\[
|D_1 g(x, y) - 0| = \left| \frac{2xy^6}{x^4 + 2x^2y^2 + y^4} \right| = \left| \frac{2xy^2}{x^2 + \frac{2x^2}{y^2} + 1} \right| \leq 2|x|y^2 \to 0 \text{ as } (x, y) \to (0,0)
\]

\[
|D_2 g(x, y) - 0| = \left| \frac{4x^4y^3 + 2x^2y^5}{x^4 + 2x^2y^2 + y^4} \right| = \left| \frac{4x^2y + 2y^3}{x^2 + 2 + \frac{y^2}{x^2}} \right| \leq 2x^2|y| + |y|^3 \to 0 \text{ as } (x, y) \to (0,0).
\]

This shows \( D_1 g(x, y) \) and \( D_2 g(x, y) \) are continuous; thus, \( g \) is continuously differentiable.

19. Suppose \( g : \mathbb{R}^2 \to \mathbb{R} \) has the property \( |g(x, y)| \leq x^2 + y^2 \) for all \( (x, y) \in \mathbb{R}^2 \). Show \( g \) has partial derivatives with respect to both \( x \) and \( y \) at \((0, 0)\).

**Solution.** Our assumption on \( g \) forces \( g(0, 0) = 0 \) and thus

\[
\left| \frac{g(t, 0) - g(0, 0)}{t} \right| \leq \left| \frac{t^2}{t} \right| = |t|, \quad \left| \frac{g(0, t) - g(0, 0)}{t} \right| \leq \left| \frac{t^2}{t} \right| = |t|.
\]

Taking limits as \( t \to 0 \) in the above expressions, we find that \( D_1 g(0, 0) = D_2 g(0, 0) = 0 \).

20. Suppose \( f : \mathbb{R}^2 \to \mathbb{R} \) has first-order partial derivatives and

\[
D_1 f(x, y) = D_2 f(x, y) = 0 \quad \text{for all } (x, y) \in \mathbb{R}^2.
\]

Show that \( f \equiv c \) for some \( c \in \mathbb{R} \); that is, \( f \) is a constant function.

**Solution.** Fix \( y_0 \in \mathbb{R} \) and consider \( g : \mathbb{R} \to \mathbb{R} \) defined by \( g(x) = f(x, y_0) \). Suppose \( g \) is nonconstant. Then \( g(a) \neq g(b) \) for some \( a < b \), and by MVT there is \( r \in (a, b) \) such that \( D_1 f(r, y_0) = \frac{g'(r)}{r} = \frac{[g(b) - g(a)]/(b - a)}{b - a} \neq 0 \), contradiction. Thus, \( g \) is constant, say \( g \equiv c \). Let \( (x, y) \in \mathbb{R}^2 \). By repeating the argument above, we find that the function \( h : \mathbb{R} \to \mathbb{R} \) defined by \( h(z) = f(x, z) \) is constant. Since \( h(y_0) = g(x) \) we must have \( h \equiv c \), and in particular \( f(x, y) = c \). Since \( (x, y) \in \mathbb{R}^2 \) was arbitrary, we can conclude \( f \equiv c \).

21. Given \( \phi, \psi : \mathbb{R}^2 \to \mathbb{R} \), a function \( f : \mathbb{R}^2 \to \mathbb{R} \) is called a **potential function** for \( \phi, \psi \) if

\[
D_1 f(x, y) = \phi(x, y) \quad \text{and} \quad D_2 f(x, y) = \psi(x, y) \quad \text{for all } (x, y) \in \mathbb{R}^2.
\]

Show that when a potential function exists for \( \phi, \psi \), it is unique up to an additive constant. Then show that if there is a potential function for \( \phi, \psi \) and \( \phi, \psi \) are continuously differentiable, then

\[
D_1 \psi(x, y) = D_2 \phi(x, y) \quad \text{for all } (x, y) \in \mathbb{R}^2.
\]
Solution. Suppose \( f \) and \( g \) are two potential functions for \( \phi, \psi \) and let \( h = f - g \). Then
\[
D_i h(x, y) = D_i f(x, y) - D_i g(x, y) = 0 \quad \text{for all } (x, y) \in \mathbb{R}^2, \ i = 1, 2,
\]
so by Problem 20, \( h \) is constant. Thus, \( f \) and \( g \) differ by a constant. The statement in (4) follows from the assumption \( D_1 f \) and \( D_2 f \) are continuously differentiable and Theorem 13.10.

22. Define \( f : \mathbb{R}^2 \to \mathbb{R} \) by
\[
f(x, y) = \begin{cases} 
\frac{x^3 y - y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\
0, & (x, y) = (0, 0).
\end{cases}
\]
Show that \( D_1 f(0, y) = -y \) for all \( y \in \mathbb{R} \) and \( D_2 f(x, 0) = x \) for all \( x \in \mathbb{R} \). Conclude that \( D_2 D_1 f(0, 0) = -1 \) but \( D_1 D_2 f(0, 0) = 1 \).

Solution. Away from \((0, 0)\) we can calculate partial derivatives of \( f \) “as usual,” i.e., without resorting to the limit definition. Thus, for \((x, y) \neq (0, 0)\),
\[
D_1 f(x, y) = \frac{(x^2 + y^2)(3x^2 y - y^3) - (x^3 y - xy^3)(2x)}{(x^2 + y^2)^2},
D_2 f(x, y) = \frac{(x^2 + y^2)(x^3 - 3xy^2) - (x^3 y - xy^3)2y}{(x^2 + y^2)^2}.
\]
Thus, for \( y \neq 0 \) and \( x \neq 0 \),
\[
D_1 f(0, y) = \frac{y^2 (-y^3)}{(y^2)^2} = -y, \quad D_2 f(x, 0) = \frac{x^2 x^3}{(x^2)^2} = x,
\]
and, from the limit definition of partial derivatives, \( D_1 f(0, 0) = 0 = D_2 f(0, 0) \). (Check this! See Problem 18 for a similar calculation.) Thus, \( D_2 D_1 f(0, y) = -1 \) and \( D_1 D_2 f(x, 0) = 1 \) for all \( x, y \in \mathbb{R} \). In particular, \( D_2 D_1 f(0, 0) = -1 \) but \( D_1 D_2 f(0, 0) = 1 \).

23. Let \( A \subset \mathbb{R}^2 \) be an open set containing \((x_0, y_0)\). Prove that there is \( r > 0 \) such that \((x, y) \in A\) whenever \(|x - x_0| < 2r \) and \(|y - y_0| < 2r\).

Solution. Pick \( \epsilon > 0 \) such that \( B_\epsilon(x_0, y_0) \subset A \). If \( 0 < r < \epsilon/(2 \sqrt{2}) \), then
\[
\|(x, y) - (x_0, y_0)\| = \sqrt{(x - x_0)^2 + (y - y_0)^2} < \sqrt{(2r)^2 + (2r)^2} = \sqrt{8r^2} = 2\sqrt{2}r < \epsilon
\]
whenever \(|x - x_0| < 2r \) and \(|y - y_0| < 2r\).
24. Suppose \( f : \mathbb{R}^n \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \) are continuously differentiable. Find a formula for \( \nabla (g \circ f)(x) \) in terms of \( \nabla f(x) \) and \( g'(f(x)) \).

**Solution.** By the definition of partial derivative and the mean value theorem,

\[
D_i(g \circ f)(x) = \lim_{t \to 0} \frac{g(f(x + te_i)) - g(f(x))}{t} = \lim_{t \to 0} g'(z_t) \frac{f(x + te_i) - f(x)}{t},
\]

where \( z_t \) is on the line segment between \( f(x) \) and \( f(x + te_i) \). Notice \( \lim_{t \to 0} g'(z_t) = g'(f(x)) \) due to continuity of \( f \) and \( g' \), and \( D_i f(x) := \lim_{t \to 0}[f(x + te_i) - f(x)]/t \). Thus,

\[
D_i(g \circ f)(x) = g'(f(x))D_i f(x), \quad i = 1, \ldots, n,
\]

that is, \( \nabla (g \circ f)(x) = \nabla f(x) g'(f(x)) \).

25. Suppose \( f : \mathbb{R}^n \to \mathbb{R} \) is such that \( D_v f(x) \) exists. Prove that \( D_{cv} f(x) = c D_v f(x) \) for any nonzero \( c \in \mathbb{R} \).

**Solution.** This follows from the computation

\[
D_{cv} f(x) = \lim_{t \to 0} \frac{f(x + tcv) - f(x)}{t} = \lim_{t \to 0} c \frac{f(x + tcv) - f(x)}{tc} = c \lim_{s \to 0} \frac{f(x + sv) - f(x)}{s} = c D_v f(x).
\]

26. Suppose \( f : \mathbb{R}^n \to \mathbb{R} \) has first-order partial derivatives and that \( x \in \mathbb{R}^n \) is a local minimizer for \( f \), that is, there is \( \epsilon > 0 \) such that

\[
f(x + h) \geq f(x) \text{ for } h \in B_\epsilon(0). \tag{5}
\]

Prove that \( \nabla f(x) = 0 \).

**Solution.** Due to the assumption (5), for \( i = 1, \ldots, n \),

\[
D_{e_i} f(x) = \lim_{t \to 0} \frac{f(x + te_i) - f(x)}{t} \geq 0,
\]

\[
D_{-e_i} f(x) = \lim_{t \to 0} \frac{f(x - te_i) - f(x)}{t} \geq 0,
\]

and by problem 25, \( D_{e_i} f(x) = -D_{-e_i} f(x) \leq 0 \). Thus \( D_{e_i} f(x) = 0 \), and so \( \nabla f(x) = 0 \).

27. Consider

\[
f(x, y, z) = xyz + x^2 + y^2.
\]
Find $\theta \in (0, 1)$ such that

$$f(1, 1, 1) - f(0, 0, 0) = D_1 f(\theta, \theta, \theta) + D_2 f(\theta, \theta, \theta) + D_3 f(\theta, \theta, \theta).$$

**Solution.** Note that

$$D_1 f(x, y, z) = yz + 2x, \quad D_2 f(x, y, z) = xz + 2y, \quad D_3 f(x, y, z) = xy.$$ 

and $f(1, 1, 1) - f(0, 0, 0) = 3$. Thus, we solve $3\theta^2 + 4\theta = 3$ to get $\theta = -\frac{2}{3} + \frac{\sqrt{13}}{3}$.

28. Define $f: \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} 
\frac{x\sqrt{x^2 + y^2}}{|y|}, & \text{if } y \neq 0, \\
0, & \text{if } y = 0.
\end{cases}$$

a) Prove $f$ is not continuous at $(0, 0)$.
b) Prove $f$ has directional derivatives in all directions at $(0, 0)$.
c) Prove for any $c \in \mathbb{R}$ there is $p$ such that 

$$\|p\| = 1 \text{ and } D_p f(0, 0) = c.$$ 

Does this contradict Corollary 13.18?

**Solution.** a) Note that $f(x, y) \equiv 1$ when $y = \frac{x^2}{\sqrt{1 - x^2}}$ and $x \neq 0$. Approaching $(0, 0)$ along this curve shows $f$ is not continuous at zero, since $f(0, 0) = 0 \neq 1$.

b) When $b \neq 0$,

$$D_{(a,b)} f(0, 0) = \lim_{t \to 0} \frac{f(ta, tb) - f(0, 0)}{t} = \lim_{t \to 0} \frac{ta\sqrt{t^2a^2 + t^2b^2}}{t|tb|} = a\sqrt{\frac{a^2}{b^2} + 1}.$$ 

Also,

$$D_{(a,0)} f(0, 0) = \lim_{t \to 0} \frac{f(ta, 0) - f(0, 0)}{t} = \lim_{t \to 0} \frac{0 - 0}{t} = 0.$$ 

Thus, $f$ has directional derivatives in every direction at $(0, 0)$.

c) We show such $p$ exists by solving the equations

$$a\sqrt{\frac{a^2}{b^2} + 1} = c, \quad a^2 + b^2 = 1 \quad \ldots \quad a = \frac{c}{\sqrt{1 + c^2}}, \quad b = \frac{1}{\sqrt{1 + c^2}}.$$ 

This does not contradict the corollary since $f$ is not continuously differentiable. (See also Problem 33 below.)
29. Suppose \( f : \mathbb{R}^n \to \mathbb{R} \) is continuously differentiable and let \( K = \{ x \in \mathbb{R}^n : \| x \| = 1 \} \). Show there is a point \( x \in K \) at which \( f|_K \) attains its smallest value. Now suppose whenever \( p \in \mathbb{R}^n \) is a unit vector, \( \langle \nabla f(p), p \rangle > 0 \). Show that then \( \| x \| < 1 \).

\[ \text{Solution.} \] \( K \) is sequentially compact and \( f \) is continuous, so \( f|_K \) attains a smallest value by the extreme value theorem (Theorem 11.22). Let \( p \in \mathbb{R}^n \) with \( \| p \| = 1 \). Then \( \langle \nabla f(p), p \rangle > 0 \) and

\[
0 = \lim_{t \to 0} \frac{f(p - tp) - f(p) - \langle \nabla f(p), -tp \rangle}{t} = \lim_{t \to 0} \frac{f(p - tp) - f(p)}{t} + \langle \nabla f(p), p \rangle.
\]

This shows that for \( t > 0 \) sufficiently close to zero, \( f(p - tp) - f(p) < 0 \). Thus, the minimum of \( f|_K \) cannot be attained at \( p \). For an \( \epsilon-\delta \) proof of this, let \( \epsilon = \langle \nabla f(p), p \rangle \) and pick \( \delta > 0 \) such that \( |t^{-1}[f(p - tp) - f(p)] + \langle \nabla f(p), p \rangle| < \epsilon \) whenever \( 0 < |t| < \delta \). Then \( f(p - tp) - f(p) < 0 \) whenever \( 0 < t < \delta \); why?

30. Prove that

\[
\lim_{(x,y) \to (0,0)} \frac{\sin(2x + 2y) - 2x - 2y}{\sqrt{x^2 + y^2}} = 0.
\]

\[ \text{Solution.} \] Let \( f(x, y) = \sin(2x + 2y) \). Then \( D_1 f(x, y) = 2 \cos(2x + 2y) = D_2 f(x, y) \) are continuous. Thus, \( f \) is continuously differentiable, so since \( D_1 f(0, 0) = D_2 f(0, 0) = 2 \), by the first order approximation theorem,

\[
\lim_{(x,y) \to (0,0)} \frac{\sin(2x + 2y) - 2x - 2y}{\| (x,y) \|} = 0.
\]

31. Suppose \( f : \mathbb{R}^2 \to \mathbb{R} \) is continuous and \( a, b \in \mathbb{R} \). Prove

\[
\lim_{(x,y) \to (0,0)} [f(x, y) - (f(0, 0) + ax + by)] = 0. \tag{6}
\]

Is it true that

\[
\lim_{(x,y) \to (0,0)} \frac{f(x, y) - [f(0, 0) + ax + by]}{\sqrt{x^2 + y^2}} = 0? \tag{7}
\]

\[ \text{Solution.} \] Since \( f \) is continuous, \( f(x, y) \to f(0, 0) = 0 \) as \( (x, y) \to (0, 0) \). Moreover, \( ax + by \to 0 \) as \( (x, y) \to (0, 0) \). This proves (6). Notice (7) is false unless \( f \) is also differentiable at \( (0,0) \) with \( D_1 f(0, 0) = a, \ D_2 f(0, 0) = b \). (See the first order approximation theorem.)

32. Define \( f : \mathbb{R}^2 \to \mathbb{R} \) by

\[
f(x, y) = \begin{cases} \sqrt{x^2 + y^2} \sin \frac{y^2}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}
\]
Show $f$ is continuous at $(0, 0)$ and has directional derivatives in every direction at $(0, 0)$, but $f$ is not differentiable at $(0, 0)$ — that is, there is no tangent plane to the graph of $f$ at $(0, 0)$.

**Solution.** Notice $|f(x, y) - f(0, 0)| = \sqrt{x^2 + y^2} \sin \frac{y^2}{x^2} \leq \sqrt{x^2 + y^2} \to 0$ as $(x, y) \to 0$. Thus, $f$ is continuous at $(0, 0)$. Since $t \mapsto \sin(t)$ is continuous for $c \in \mathbb{R}$, when $a \neq 0$,

$$D_{(a,b)}f(0,0) = \lim_{t \to 0} \frac{f(ta, tb) - f(0, 0)}{t}$$

$$= \lim_{t \to 0} \frac{\sqrt{t^2a^2 + t^2b^2} \sin \frac{t^2b^2}{ta}}{t} = \lim_{t \to 0} \sqrt{a^2 + b^2} \left| \sin \frac{tb^2}{a} \right| = 0.$$

Also, $D_{(0,b)}f(0,0) = \lim_{t \to 0} \frac{f(0,tb) - f(0,0)}{t} = \lim_{t \to 0} \frac{0-0}{t} = 0$. Thus, $f$ has directional derivatives in every direction at $(0, 0)$. Note that $D_1f(0,0) = D_2f(0,0) = 0$ but the limit

$$\lim_{(x,y) \to (0,0)} \frac{f(x, y) - f(0, 0) - 0x - 0y}{\| (x, y) \|} = \lim_{(x,y) \to (0,0)} \frac{\sin \frac{y^2}{x}}{x}$$

does not exist (e.g., take $x \equiv y^3$ and let $y \to 0$). Thus, $f$ is not differentiable at $(0, 0)$.

33. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous, that is, $f(0) = 0$ and $f(tx) = tf(x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Prove that if $f$ is differentiable, then $f(x) = \langle \nabla f(0), x \rangle$, in particular, $f$ is linear.\footnote{$f$ is linear if it is homogeneous and additive: $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^n$.} Thus, if $f$ is homogeneous and not linear, it cannot be differentiable.

**Solution.** Let $f$ be differentiable and homogeneous, and $u \in \mathbb{R}^n$ a unit vector. Then

$$0 = \lim_{t \to 0^+} \frac{f(tu) - f(0) - \langle \nabla f(0), tu \rangle}{\| tu \|} = \lim_{t \to 0^+} \frac{tf(u) - t\langle \nabla f(0), u \rangle}{t\| u \|} = f(u) - \langle \nabla f(0), u \rangle.$$

This shows that $f(u) = \langle \nabla f(0), u \rangle$ and thus for $t \in \mathbb{R}$,

$$f(tu) = tf(u) = t\langle \nabla f(0), u \rangle = \langle \nabla f(0), tu \rangle.$$

Any nonzero $x \in \mathbb{R}^n$ is a scalar times a unit vector, $x = \| x \| \frac{x}{\| x \|}$. Thus, we are done.

34. Let $f : \mathbb{R}^n \to \mathbb{R}$ be $k$ times continuously differentiable, let $h \in \mathbb{R}^n$ and define $\phi(t) = f(x + th)$ for $t \in \mathbb{R}$. Prove that for all $s \in \mathbb{R}$,

$$\phi^{(k)}(s) = D_h^k f(x + sh), \quad D_h^k f = D_h D_h \ldots D_h f \quad \text{for } k \text{ times}.$$

Write formulas for $\phi^{(k)}(s) = D_h^k f(x + sh)$ when $k = 1$ and $k = 2$.\footnote{$f$ is linear if it is homogeneous and additive: $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^n$.}
Solution. For \( k = 1 \) this follows from the computation
\[
\phi'(s) = \lim_{t \to 0} \frac{\phi(s + t) - \phi(s)}{t} = \lim_{t \to 0} \frac{f(x + (s + t)h) - f(x + sh)}{t} = \lim_{t \to 0} \frac{f((x + sh) + th) - f(x + sh)}{t} = D_h f(x + sh).
\]
The general case follows from induction: if \( \phi^{(k-1)}(s) = D^{k-1}_h f(x + sh) \) for all \( s \in \mathbb{R} \), then
\[
\phi^{(k)}(s) = \lim_{t \to 0} \frac{\phi^{(k-1)}(s + t) - \phi^{(k-1)}(s)}{t} = \lim_{t \to 0} \frac{D^{k-1}_h f(x + (s + t)h) - D^{k-1}_h f(x + sh)}{t} = \lim_{t \to 0} \frac{D^{k-1}_h f((x + sh) + th) - D^{k-1}_h f(x + sh)}{t} = D_h D^{k-1}_h f(x + sh) = D^k_h f(x + sh).
\]
When \( k = 1 \),
\[
\phi'(s) = D_h f(x + sh) = \sum_{i=1}^{n} h_i D_i f(x + sh) = \langle \nabla f(x + sh), h \rangle.
\]
When \( k = 2 \),
\[
\phi''(s) = D^2_h f(x + sh) = D_h \left( \sum_{j=1}^{n} h_j D_j f(x + sh) \right) = \sum_{i=1}^{n} h_i D_i \left( \sum_{j=1}^{n} h_j D_j f(x + sh) \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} h_i h_j D_i D_j f(x + sh) = \langle \nabla^2 f(x + sh) h, h \rangle,
\]
where the last step uses the fact that \( \langle Ah, h \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} h_i h_j a_{ij} \) when \( A \) is \( n \times n \).

35. Define \( f : \mathbb{R}^2 \to \mathbb{R} \) by \( f(x, y) = e^{xy} + x^2 + 2xy \). Define \( \phi(t) = f(2t, 3t) \) for \( t \in \mathbb{R} \) and compute \( \phi''(0) \) in the following two ways:

i) By using single-variable theory, i.e., differentiating the single-variable function \( \phi \);

ii) By thinking of \( \phi''(0) \) as a directional derivative of \( f \) in the direction of \( h = (2, 3) \).

Solution. i) Note that \( \phi(t) = e^{6t^2} + 16t^2 \), so direct computation gives \( \phi''(0) = 44 \).

ii) By\(^3\) Problem 34, \( \phi''(0) = \langle \nabla^2 f(0,0) h, h \rangle \) where \( h = (2, 3) \). We compute
\[
\nabla^2 f(0,0) = \begin{pmatrix} 2 & 3 \\ 3 & 0 \end{pmatrix}
\]
\(^3\)or formula (14.11) in the text
and $\langle \nabla^2 f(0,0)h,h \rangle = \langle (13,6), (2,3) \rangle = 44$.

36. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be twice continuously differentiable. Suppose that $\nabla f(0,0) = (0,0)$ and that for some $h \in \mathbb{R}^2$, $\langle \nabla^2 f(0,0)h,h \rangle > 0$. Using single variable theory, prove there is $r > 0$ such that

$$f(th) > f(0,0), \quad 0 < |t| < r.$$ 

Solution. Define $\phi(t) = f(th)$. By Taylor’s theorem in 1D,

$$\phi(t) = \phi(0) + \phi'(0)t + \frac{1}{2}\phi''(0)t^2 + o(t^2)$$

where $\lim_{t \to 0} o(t^2)/t^2 = 0$. By Problem 34, this can be rewritten

$$f(th) = f(0,0) + \langle \nabla f(0,0), h \rangle t + \frac{1}{2}\langle \nabla^2 f(0,0)h,h \rangle t^2 + o(t^2).$$

Since $\nabla f(0,0) = (0,0)$, $\langle \nabla f(0,0), h \rangle = 0$ and we can rewrite again to get

$$\frac{f(th) - f(0,0)}{t^2} = \frac{1}{2}\langle \nabla^2 f(0,0)h,h \rangle + \frac{o(t^2)}{t^2}.$$ 

Since $\lim_{t \to 0} o(t^2)/t^2 = 0$ and $\langle \nabla^2 f(0,0)h,h \rangle > 0$, the RHS of this equation is positive for sufficiently small $t$. Thus, $f(th) > f(0,0)$ for sufficiently small $t$. More precisely, let $\epsilon = \frac{1}{2}\langle \nabla^2 f(0,0)h,h \rangle$ and choose $\delta > 0$ such that $|o(t^2)/t^2| < \epsilon$ whenever $0 < |t| < \delta$. Then $f(th) > f(0,0)$ for $0 < |t| < \delta$, so we can take $r = \delta$.

37. In Problem 36, suppose additionally that $\langle \nabla^2 f(0,0)h,h \rangle > 0$ for all nonzero $h \in \mathbb{R}^n$. Is this enough to directly conclude the origin is a local minimum of $f$?

Solution. We would have to show that $f(th) > f(0,0)$ for all $0 < |t| < r$, where $r$ does not depend on $h$. The trouble with using Problem 36 here is that we had $r = r_h$ that depends on $h$. To get an $r$ which is independent of $h$, we could try $r = \inf_h r_h$, but then we might get $r = 0$.

Note that in this situation the origin is indeed a local minimizer – see for instance Theorem 14.22. But this conclusion cannot be reached directly from Problem 36.

38. Let $a, b, c \in \mathbb{R}$ with $a \neq 0$, and define $p(t) = at^2 + 2bt + c$. Show that $p(t) > 0$ for all $t$ if and only if $a > 0$ and $ac - b^2 > 0$. Show also that $p(t) < 0$ for all $t$ if and only if $a < 0$ and $ac - b^2 > 0$. 

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Solution. The quadratic equation shows that \( p(t) = 0 \) when
\[
t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]
Suppose \( ac - b^2 > 0 \). Then \( p(t) = 0 \) has no solutions. So by the intermediate value theorem, either \( p(t) > 0 \) for all \( t \) or \( p(t) < 0 \) for all \( t \). To finish the proof, notice that \( \lim_{t \to \infty} p(t)/t^2 = a \). If \( a > 0 \) then \( p(t) > 0 \) for sufficiently large \( t \), hence for all \( t \); while if \( a < 0 \) then \( p(t) < 0 \) for sufficiently large \( t \), hence for all \( t \).

39. Find \( 2 \times 2 \) matrices associated with the quadratic functions \( h(x, y) = x^2 - y^2 \) and \( g(x, y) = x^2 + 8xy + y^2 \).

Solution. They are
\[
A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}.
\]
With \( z = (x, y) \) we have \( h(z) = \langle Az, z \rangle \) and \( g(z) = \langle Bz, z \rangle \).

40. Define \( Q: \mathbb{R} \to \mathbb{R} \) by \( Q(x) = x^4 \). Show that there is no \( c > 0 \) such that \( Q(x) \geq cx^2 \) for all \( x \neq 0 \). Explain why this does not contradict Proposition 14.16.

Solution. For any \( c > 0 \), if \( x^2 < c \) then \( Q(x) = x^4 = x^2 x^2 < cx^2 \). This does not contradict Proposition 14.16 because \( Q \) is not a quadratic function: it cannot be written in the form \( Q(x) = \langle Ax, x \rangle \) for a matrix \( A \). (Such a matrix \( A \) would have to \( 1 \times 1 \), and so \( \langle Ax, x \rangle \) would be a scalar constant times \( x^2 \).)

41. Show that the point \((-1, 1)\) is the minimizer of the function \( f: \mathbb{R}^2 \to \mathbb{R} \) defined by
\[
f(x, y) = (2x + 3y)^2 + (x + y - 1)^2 + (x + 2y - 2)^2.
\]

Solution. Notice \( \nabla f(x, y) = (12x + 18y - 6, 18x + 28y - 10) \) and
\[
\nabla^2 f(x, y) \equiv \begin{pmatrix} 12 & 18 \\ 18 & 28 \end{pmatrix}.
\]
In particular, \( \nabla f(x, y) = 0 \iff (x, y) = (-1, 1) \), and since \( 12 > 0, 12 \cdot 28 - 18 \cdot 18 = 12 > 0 \), \( \nabla^2 f(x, y) \) is positive definite for all \( (x, y) \). Thus, \( f \) has a local minimum at \((1, 1)\). We argue this must be a global minimum. Suppose to the contrary that \( f(r, s) < f(-1, 1) \). Let \( L(t) = (-1, 1) + t(r, s) \) and define \( g = f \circ L \). Since \( \nabla f(-1, 1) = (0, 0) \) and \( \nabla^2 f(x, y) \) is always positive definite, we have \( g'(0) = 0 \) and \( g''(t) > 0 \) for all \( t \in \mathbb{R} \). (To see this,
apply Theorem 14.12 or Problem 34 above.) Thus, 0 is a global minimum for $g$. But $g(1) = f(r, s) < f(-1, 1) = g(0)$, contradiction.

42. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable. Let $x \in \mathbb{R}^n$ be such that $\nabla f(x) = 0$. Suppose there are $u, v \in \mathbb{R}^n$ such that

$$\langle \nabla^2 f(x) u, u \rangle > 0, \quad \langle \nabla^2 f(x) v, v \rangle < 0.$$ 

Prove that $x$ is neither a local maximum nor local minimum of $f$.

Solution. By the second order approximation theorem\(^4\)

$$f(x + tu) - f(x) = \langle \nabla f(x), tu \rangle + \frac{1}{2} \langle \nabla^2 f(x) tu, tu \rangle + o((tu)^2) = \frac{t^2}{2} \langle \nabla^2 f(x) u, u \rangle + o((tu)^2).$$

where $\lim_{t \to 0} o((tu)^2)/\|tu\|^2 = 0$. Rearranging, we find that

$$\frac{f(x + tu) - f(x)}{t^2} = \frac{1}{2} \langle \nabla^2 f(x) u, u \rangle + \frac{o((tu)^2)}{t^2}$$

where $\lim_{t \to 0} o((tu)^2)/t^2 = 0$. We can conclude that the LHS above is positive for sufficiently small $t$. An analogous argument shows the LHS of

$$\frac{f(x + tv) - f(x)}{t^2} = \frac{1}{2} \langle \nabla^2 f(x) v, v \rangle + \frac{o((tv)^2)}{t^2}$$

is negative for sufficiently small $t$. (We will not give $\epsilon$-$\delta$ delta proofs here; see however similar proofs in Problems 29, 36 and 43 below.) Thus, $f$ has a “saddle point” at $x$.

43. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable, $\nabla f(x) = 0$, and $\nabla^2 f(x)$ is positive definite. Prove that there exist $c > 0$ and $\delta > 0$ such that

$$f(x + h) - f(x) \geq c \|h\|^2 \quad \text{if} \quad \|h\| < \delta.$$

Solution. By the second order approximation theorem,

$$f(x + h) - f(x) = \langle \nabla f(x), h \rangle + \frac{1}{2} \langle \nabla^2 f(x) h, h \rangle + o(h^2) = \frac{1}{2} \langle \nabla^2 f(x) h, h \rangle + o(h^2).$$

Thus,

$$\frac{f(x + h) - f(x)}{\|h\|^2} = \frac{1}{2} \left\langle \nabla^2 f(x) \frac{h}{\|h\|^2}, \frac{h}{\|h\|^2} \right\rangle + \frac{o(h^2)}{\|h\|^2}.$$
Let
\[ c = \min_{u \in \mathbb{R}^n : \|u\| = 1} \frac{1}{4} \langle \nabla^2 f(x) u, u \rangle. \]

Notice the minimum above is attained at some \( u \) on the unit sphere \( \{ u \in \mathbb{R}^n : \|u\| = 1 \} \), since the unit sphere is sequentially compact. Since \( \nabla^2 f(x) \) is positive definite, this means that \( c > 0 \). Now choose \( \delta > 0 \) such that \( \frac{1}{\|h\|^2} |o(h^2)/\|h\|^2| < c \) whenever \( 0 < \|h\| < \delta \).

Then if \( 0 < \|h\| < \delta \), we have
\[
\frac{f(x+h) - f(x)}{\|h\|^2} = \frac{1}{2} \left\langle \nabla^2 f(x) \frac{h}{\|h\|}, \frac{h}{\|h\|} \right\rangle + \frac{o(h^2)}{\|h\|^2} \geq 2c - c = c.
\]

44. Suppose \( f : \mathbb{R}^n \to \mathbb{R} \) is twice continuously differentiable. Suppose that \( \nabla f(x) = 0 \) and \( \nabla^2 f(x) \) is the matrix of all zeros. Show that \( x \) could be a local maximum, a local minimum, or neither.

**Solution.** Let \( n = 2 \) and consider the functions \( f(x, y) = -x^4 - y^4 \), \( f(x, y) = x^4 - y^4 \) and \( f(x, y) = x^4 + y^4 \); these functions all satisfy the assumptions above and they have, respectively, a local maximum, saddle point, and local minimum at \((0, 0)\).

45. Which of the following functions \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) are linear?
   i) \( f(x, y) = (-y, e^x) \)
   ii) \( f(x, y) = (x - y^2, 2y) \)
   iii) \( f(x, y) = 17(x, y) \)

**Solution.** In i), note that \( 2f(1, 0) = (0, 2e) \neq (0, e^2) = f(2, 0) \), and in ii), notice \( 2f(0, 1) = (-2, 4) \neq (-4, 4) = f(0, 2) \). So the functions in i) and ii) are not linear. The function in iii) is linear with matrix
\[
\begin{pmatrix} 17 & 0 \\ 0 & 17 \end{pmatrix}.
\]

46. Show there is no linear mapping \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) having the property that \( T(1, 1) = (4, 0) \) and \( T(-2, -2) = (0, 1) \).

**Solution.** Suppose \( T \) were linear with \( T(1, 1) = (4, 0) \) and \( T(-2, -2) = (0, 1) \). Then \( (0, 0) = T(0, 0) = T(2(1, 1) + (-2, -2)) = 2T(1, 1) + T(-2, -2) = (8, 0) + (0, 1) = (8, 1) \), contradiction.
47. For a point \((x, y)\) in the plane \(\mathbb{R}^2\), define \(T(x, y)\) to be the point on the line \(\ell = \{(x, y) \in \mathbb{R}^2 : y = 2x\}\) that is closest to \((x, y)\). Show that \(T : \mathbb{R}^2 \to \mathbb{R}^2\) is linear and find its associated matrix.

**Solution.** Observe that \(T(x, y) = (r, 2r)\) where \(r\) is the minimizer of
\[
f(r) = (x - r)^2 + (y - 2r)^2.
\]
Notice \(f'(r) = 10r - 2x - 4y = 0\) if and only if \(r = x/5 + 2y/5\). Since \(f''(r) = 10 > 0\) for all \(r\), this is the unique global minimizer. Thus, \(T\) is linear with matrix
\[
\begin{pmatrix}
\frac{1}{5} & \frac{2}{5} \\
\frac{2}{5} & \frac{4}{5}
\end{pmatrix}.
\]

48. Suppose \(A\) is an \(m \times n\) matrix. Define \(f : \mathbb{R}^n \to \mathbb{R}^m\) by
\[
f(x) = Ax.
\]
Prove that \(Df(x) = A\) for all \(x \in \mathbb{R}^n\).

**Solution.** This follows from uniqueness of the derivative: since for every \(x \in \mathbb{R}^n\),
\[
\lim_{h \to 0} \frac{f(x + h) - f(x) - Ah}{\|h\|} = \lim_{h \to 0} \frac{A(x + h) - Ax - Ah}{\|h\|} = \lim_{h \to 0} \frac{0}{\|h\|} = 0,
\]
we must have \(Df(x) = A\) for every \(x \in \mathbb{R}^n\).

49. Give a proof of the first order approximation theorem based on the mean value theorem.

**Solution.** Suppose \(f : \mathbb{R}^n \to \mathbb{R}^m\) is continuously differentiable with component functions \(f_1, \ldots, f_m\). By the mean value theorem,
\[
f_i(x + h) - f_i(x) = \langle \nabla f_i(z^{(i)}), h \rangle,
\]
where \(z^{(i)}\) is on the line segment between \(x\) and \(x + h\). Let \(x\) be fixed and let \(B(h)\) be the matrix whose \(i\)th row is \(\nabla f_i(z^{(i)})\). Combining the component equations above,
\[
f(x + h) - f(x) = B(h)h.
\]
Thus,
\[
\lim_{h \to 0} \frac{f(x + h) - f(x) - Df(x)h}{\|h\|} = \lim_{h \to 0} \left[ \frac{f(x + h) - f(x) - B(h)h}{\|h\|} + \frac{B(h)h - Df(x)h}{\|h\|} \right] = \lim_{h \to 0} \frac{B(h)h - Df(x)h}{\|h\|} = \lim_{h \to 0} [B(h) - Df(x)] \frac{h}{\|h\|}.
\]
Note \( \lim_{h \to 0} B(h) - Df(x) = 0 \) by continuity of partial derivatives of \( f \). Moreover, \( h/\|h\| \) is bounded. Thus, the limit above is 0. To see why, note that the generalized Cauchy-Schwartz inequality (Theorem 14.13) yields

\[
\lim_{h \to 0} \frac{\|B(h) - Df(x)\|}{\|h\|} \leq \lim_{h \to 0} \|B(h) - Df(x)\| = 0.
\]

50. Suppose \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) is continuously differentiable with \( f(x, y) = (\psi(x, y), \phi(x, y)) \). Define \( g : \mathbb{R}^2 \to \mathbb{R}^2 \) by \( g(x, y) = \frac{1}{2}[\psi(x, y)^2 + \phi(x, y)^2] \). Show that \( Dg(x_0, y_0) = f(x_0, y_0)Df(x_0, y_0) \). Use this to show that if \( (x_0, y_0) \) is a minimizer of \( g : \mathbb{R}^2 \to \mathbb{R} \) and \( Df(x_0, y_0) \) is invertible, then \( f(x_0, y_0) = 0 \).

Solution. Let \( h(x, y) = \frac{1}{2}[x^2 + y^2] \). Then \( g(x, y) = (h \circ f)(x, y) \). By the chain rule,

\[
Dg(x, y) = Dh(f(x, y))Df(x, y) = (\psi(x, y) \phi(x, y)) \begin{pmatrix} D_1 \psi(x, y) & D_2 \psi(x, y) \\ D_1 \phi(x, y) & D_2 \phi(x, y) \end{pmatrix} = f(x, y)Df(x, y).
\]

The result follows when \( x = x_0, y = y_0 \) by multiplying both sides above on the right by \( Df(x_0, y_0)^{-1} \) and using the fact that \( Dg(x_0, y_0) = 0 \) since \( (x_0, y_0) \) is a minimizer of \( g \).

51. Suppose \( \psi : \mathbb{R}^2 \to \mathbb{R} \) is continuously differentiable and define \( g : \mathbb{R}^2 \to \mathbb{R} \) by

\[
g(s, t) = \psi(s^2t, s).
\]

Find \( \partial g/\partial s(s, t) \) and \( \partial g/\partial t(s, t) \).

Solution. By the chain rule,

\[
\frac{\partial g}{\partial s} = \frac{\partial g}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial g}{\partial y} \frac{\partial y}{\partial s}, \quad \frac{\partial g}{\partial t} = \frac{\partial g}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial g}{\partial y} \frac{\partial y}{\partial t}.
\]

That is,

\[
\frac{\partial g}{\partial s}(s, t) = D_1 \psi(s^2t, s)2st + D_2 \psi(s^2t, s),
\]

\[
\frac{\partial g}{\partial t}(s, t) = D_1 \psi(s^2t, s)s^2.
\]

52. Suppose \( g, h : \mathbb{R}^2 \) have continuous second derivatives and define

\[
u(s, t) = g(s - t) + h(s + t).
\]
Prove that
\[ \frac{\partial^2 u}{\partial t^2}(s,t) - \frac{\partial^2 u}{\partial s^2}(s,t) = 0 \]
for all \((s,t) \in \mathbb{R}^2\).

**Solution.** By the chain rule,
\[ \frac{\partial u(s,t)}{\partial s} = g'(s-t) + h'(s+t), \quad \frac{\partial u(s,t)}{\partial t} = -g'(s-t) + h'(s+t) \]
and
\[ \frac{\partial^2 u}{\partial s}(s,t) = g''(s-t) + h''(s+t), \quad \frac{\partial^2 u}{\partial s}(s,t) = -(-g''(s-t)) + h''(s+t). \]
The result follows.

53. Let \(U, V \subset \mathbb{R}\) be open, suppose \(f : U \to V\) is differentiable and bijective, and let \(a \in U\) be such that \(f'(a) = 0\). Prove that \(f^{-1} : V \to U\) is not differentiable at \(b = f(a)\).

**Solution.** Suppose \(g := f^{-1}\) is differentiable at \(b\). Then by the chain rule,
\[(g \circ f)'(a) = g'(b)f'(a) = 0.\]
But \((g \circ f)(x) = x\) for all \(x \in U\), so that \((g \circ f)'(a) = 1\), contradiction.

54. For the following mappings \(f, g : \mathbb{R}^2 \to \mathbb{R}^2\), apply the inverse function theorem at the point \((0,0)\) and calculate the partial derivatives of the components of the inverse mapping at the points \(f(0,0)\) and \(g(0,0)\).

a) \(f(x, y) = (x + x^2 + e^{x^2 y^2}, -x + y + \sin(xy))\), b) \(g(x, y) = (e^{x+y}, e^{x-y})\).

**Solution.** Since
\[ Df(0,0) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad Dg(0,0) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \]
and \(f(0,0) = (1,0)\) and \(g(0,0) = (1,1)\), by the inverse function theorem
\[ Df^{-1}(1,0) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad Dg^{-1}(1,1) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}. \]
The first (resp. second) row of \(Df^{-1}(1,0)\) contains the partial derivatives of the first (resp. second) component of \(f^{-1}\) at the point \((1,0)\), and similarly for \(g^{-1}\).
55. Define \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) by \( f(x, y) = (e^x \cos y, e^x \sin y) \). Show that the inverse function theorem applies at every point \((x, y) \in \mathbb{R}^2\), but that \( f \) is not one-to-one. Does the latter statement contradict the former?

*Solution.* Observe that
\[
Df(x, y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}
\]
and so \( \det Df(x, y) = e^x \neq 0 \) for all \((x, y) \in \mathbb{R}^2\). Thus, the inverse function applies at all \((x, y) \in \mathbb{R}^2\). However, \( f \) is not one-to-one since \( f(x, y) = f(x, y + 2\pi) \). There is no contradiction here: a function can be locally invertible at every point without having a global inverse.

56. For a pair of real numbers \( a, b \), consider the system of nonlinear equations
\[
\begin{align*}
x + x^2 \cos y + xy e^{3y^2} &= a, \\
y + x^5 + y^3 - x^2 \cos(xy) &= b.
\end{align*}
\]
(8)

Use the inverse function theorem to show that there is some positive number \( r \) such that if \( a^2 + b^2 < r^2 \), then this system of equations has at least one solution.

*Solution.* Let \( f(x, y) = (x + x^2 \cos y + xy e^{3y^2}, y + x^5 + y^3 - x^2 \cos(xy)) \). Then
\[
Df(x, y) = \begin{pmatrix} 1 + 2x \cos y + ye^{3y^2} + 3x^3 ye^{3y^2} & -x^2 \sin y + xe^{3y^2} + 2xy^2 e^{3y^2} \\ 5x^4 - 2x \cos(xy) + x^2 y \sin(xy) & 1 + 3y^2 + x^2 \sin(xy) \end{pmatrix}
\]
and so
\[
Df(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
which is nonsingular. So by the inverse function theorem, there are neighborhoods \( U \) of \((0, 0)\) and \( V \) of \( f(0, 0) = (0, 0) \) such that \( f : U \to V \) is invertible. Thus, the system of equations can be solved when \((a, b) \in V \) (uniquely, if one considers only \((x, y) \in U\)). Since \( V \) is open we may pick \( r > 0 \) such that \( B_r(0, 0) \subset V \), which completes the proof.

57. Suppose \( \psi : \mathbb{R}^2 \to \mathbb{R} \) is continuously differentiable and define \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) by
\[
f(x, y) = (\psi(x, y), -\psi(x, y)).
\]

Explain both geometrically and analytically why the hypotheses of the inverse function theorem must fail for \( f \) at every point \((x, y) \in \mathbb{R}^2\).

*Solution.* The analytic explanation is that
\[
\det Df(x, y) = \begin{vmatrix} D_1 \psi(x, y) & D_2 \psi(x, y) \\ -D_1 \psi(x, y) & -D_2 \psi(x, y) \end{vmatrix} = 0
\]
for every \((x, y)\). The geometric explanation is that \(\text{Im } f\) is the line \(y = x\), which is one-dimensional, and an injective continuous function defined on the plane cannot have a 1-dimensional image.

58. Define \(f(x) = x^n\) for \(x \in \mathbb{R}\). Prove that \(f\) is bijective if \(n \geq 1\) is odd. Then show that \(f\) is not stable if \(n > 1\).

\textit{Solution.} Let \(n \geq 1\) be odd. To see that \(f\) is onto, let \(y \in \mathbb{R}\). By the intermediate value theorem there is \(x\) between 0 and \(\sqrt[y]{y}\) such that \(f(x) = y\). To see that \(f\) is one-to-one, suppose \(f(a) = f(b) > 0\) for \(a \neq b\). Then \(a, b > 0\) since \(n\) is odd, and by the mean value theorem there is \(c\) between \(a\) and \(b\) such that \(f'(c) = nc^{n-1} = 0\), a contradiction since \(c > \min\{a, b\} > 0\). Similarly, \(f(a) = f(b) < 0\) only if \(a = b\). The other possibility is that \(f(a) = f(b) = 0\), but \(f(x) = x^n = 0\) only when \(x = 0\). To see that \(f\) is not stable when \(n > 1\), we use the difference of powers formula

\[ f(x) - f(y) = x^n - y^n = (x - y) \sum_{i=0}^{n-1} x^{n-1-i} y^i. \]

Suppose \(f\) were stable with constant \(c\). If \(y = 2x\) where \(0 < x < \sqrt[n]{c/(2n)}\), then

\[ \left| \frac{f(x) - f(y)}{x - y} \right| = \left| \sum_{i=0}^{n-1} x^{n-1-i} y^i \right| = 2n x^{n-1} < c, \]

contradiction.

59. Let \(f : (a, b) \to \mathbb{R}\) be differentiable. Prove that \(f\) is stable with stability constant \(c\) if and only if \(|f'(x)| \geq c\) for all \(x \in (a, b)\).

\textit{Solution.} Suppose that \(|f'(x)| \geq c\) for all \(x \in (a, b)\). If there were \(x, y \in (a, b)\) such that

\[ |f(x) - f(y)| < c|x - y| \]

then by the mean value theorem there would be a point \(r\) between \(x\) and \(y\) such that \(|f'(r)| = |f(x) - f(y)|/|x - y| < c\), contradiction. Thus, \(f\) must be stable. Conversely, suppose \(f\) is stable with constant \(c\). Then \(|f(x) - f(y)| \geq c|x - y|\) for all \(x, y \in (a, b)\) and so

\[ |f'(y)| = \lim_{x \to y} \frac{f(x) - f(y)}{x - y} \geq c. \]

60. Define \(g(x, y) = (x^2, y)\) for \((x, y) \in \mathbb{R}^2\). Show there is no neighborhood of \((0, 0)\) such that \(g : \mathbb{R}^2 \to \mathbb{R}^2\) is stable.
Suppose \( g \) were stable with stability constant \( c \). Then for \( x \in \mathbb{R} \) such that \( 0 < |x| < c \),

\[
\frac{\|g(x,0) - g(0,0)\|}{\|(x,0) - (0,0)\|} = \frac{x^2}{|x|} = |x| < c,
\]

contradiction.

61. Is the sum of stable mappings also a stable mapping?

Solution. No. For example, \( f(x) = x \) and \( g(x) = -x \), defined for \( x \in \mathbb{R} \), are both stable, but \( f + g \equiv 0 \) is not.

62. Let \( U \subset \mathbb{R}^n \) be open and suppose \( f : U \to \mathbb{R}^n \) is continuously differentiable and stable. Prove for each \( x \in U \), \( Df(x) \) is nonsingular.

Solution. Let \( x \in U \). By the first order approximation theorem,

\[
f(x + h) - f(x) = Df(x)h + o(h).
\]

Suppose \( f \) has stability constant \( c \). Then

\[
\|Df(x)h\| \geq \|Df(x)h + o(h)\| - \|o(h)\|
\]

\[
= \|f(x + h) - f(x)\| - \|o(h)\| \geq c\|h\| - \|o(h)\|.
\]

Thus,

\[
\left| \frac{Df(x)}{\|h\|} \right| \geq c - \frac{\|o(h)\|}{\|h\|} \to c \quad \text{as } h \to 0.
\]

This shows that \( Df(x)u \neq 0 \) for all unit vectors \( u \). It follows that \( Df(x) \) is nonsingular.

63. For a point \((\rho, \theta, \phi) \in \mathbb{R}^3\), define

\[
f(\rho, \theta, \phi) = (\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi).
\]

At what points does the inverse function apply to \( f \)?

Solution. Since

\[
\det Df(\rho, \theta, \phi) = \begin{vmatrix}
\sin \phi \cos \theta & -\rho \sin \phi \sin \theta & \rho \cos \phi \cos \theta \\
\sin \phi \sin \theta & \rho \sin \phi \cos \theta & \rho \cos \phi \sin \theta \\
\cos \phi & 0 & -\rho \sin \phi \\
\end{vmatrix}
= \cos \phi \begin{vmatrix}
-\rho \sin \phi \sin \theta & \rho \cos \phi \cos \theta \\
\rho \sin \phi \cos \theta & \rho \cos \phi \sin \theta \\
\end{vmatrix}
= -\rho^2 \sin \phi \cos^2 \phi - \rho^2 \sin^3 \phi = -\rho^2 \sin \phi,
\]

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the inverse function theorem applies when \( \rho \neq 0 \) and \( \phi \) is not an integer multiple of \( \pi \).

64. Define \( f(x) = x^2 \) for \( x \in \mathbb{R} \).

   a) Show that if \( U \subset \mathbb{R} \) is open and \( 0 \notin U \), then \( f(U) \) is open.

   b) Show that if \( U \subset \mathbb{R} \) is open and \( 0 \in U \), then \( f(U) \) is not open.

**Solution.**

a) Any open set \( U \subset \mathbb{R} \) is a union of open intervals. Thus, we only have to show that if \( U = (a, b) \subset \mathbb{R} \) and \( 0 \notin (a, b) \), then \( f(U) \) is open. If \( 0 < a < b \) and \( U = (a, b) \), then \( f(U) = (f(a), f(b)) \), while if \( a < b < 0 \) and \( U = (b, a) \), then \( f(U) = (f(b), f(a)) \).

b) Let \( U \subset \mathbb{R} \) with \( 0 \in U \). Then \( 0 \in f(U) \) but for every \( \epsilon > 0 \), \((-\epsilon, \epsilon) \notin f(U) \) since the image of \( f \) contains no negative numbers. Thus, \( f(U) \) is not open.

65. Give an example of a continuously differentiable mapping \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) with the property that there is no open subset \( U \subset \mathbb{R}^n \) such that \( f(U) \) is open.

**Solution.** Let \( f(x) \equiv 0 \). \( f \) is continuously differentiable (since it is linear), but for every open \( U \subset \mathbb{R}^n \), \( f(U) = \{0\} \) is not open.

66. Consider the equation \( x^2 + y^2 = 0 \) for \( (x, y) \in \mathbb{R}^2 \). Show that the assumptions of Dini’s Theorem do not hold at the point \((0, 0)\). Explain geometrically why the conclusion of Dini’s theorem does not hold at \((0, 0)\).

**Solution.** Let \( f(x, y) = x^2 + y^2 \). Since \( D_1 f(0, 0) = D_2 f(0, 0) = 0 \), the assumptions of Dini’s theorem do not hold at \((0, 0)\). Note that the solution set for \( x^2 + y^2 = 0 \) consists only of the point \((0, 0)\). Thus, \( y \) cannot be written as a function of \( x \) (or vice-versa) on an open set containing 0.

67. Consider the equation

\[
e^{2x-y} + \cos(x^2 + xy) - 2 - 2y = 0, \quad (x, y) \in \mathbb{R}^2.
\]

Does the set of solutions of this equation in a neighborhood of the solution \((0, 0)\) implicitly define \( x \) as a function of \( y \) or vice-versa? If so, compute the derivative of this function(s) at the point 0.

**Solution.** Let \( f(x, y) = e^{2x-y} + \cos(x^2 + xy) - 2 - 2y \) and note that

\[
\nabla f(x, y) = (2e^{2x-y} - (2x + y) \sin(x^2 + xy), -e^{2x-y} - x \sin(x^2 + xy) - 2).
\]
Thus, $D_1 f(0,0) = 2 \neq 0$ and $D_2 f(0,0) = -3 \neq 0$. So the set of solutions defines $x$ as a function of $y$, and vice-versa, in a neighborhood of $(0,0)$. Writing $\psi$ and $\phi$ for these functions, respectively, we have

\begin{align*}
D_1 f(\psi(x), x) \psi'(x) + D_2 f(\psi(x), x) &= 0, \\
D_1 f(x, \phi(x)) + D_2 f(x, \phi(x)) \phi'(x) &= 0,
\end{align*}

for $x$ in a neighborhood of 0. Thus,

\begin{align*}
\psi'(0) &= -\frac{D_2 f(0,0)}{D_1 f(0,0)} = \frac{3}{2}, \\
\phi'(0) &= -\frac{D_1 f(0,0)}{D_2 f(0,0)} = \frac{2}{3}.
\end{align*}

68. Let $O \subset \mathbb{R}^2$ be open and $f : O \to \mathbb{R}$ continuously differentiable. Suppose that $f(a,b) = 0$ and $\nabla f(a,b) \neq (0,0)$. Show that $\nabla f(a,b)$ is orthogonal to the tangent line at $(a,b)$ of the implicitly defined function.

*Solution.* Since $\nabla f(a,b) \neq (0,0)$, we can assume without loss of generality that $D_2 f(a,b) \neq 0$. So by Dini’s theorem, in a neighborhood of $(a,b)$ the equation $f(x,y) = 0$ implicitly defines $y = \phi(x)$ and moreover

\[ D_1 f(x, \phi(x)) + D_2 f(x, \phi(x)) \phi'(x) = 0. \]

Setting $x = a$, this rewrites as

\[ \langle \nabla f(a,b), (1, \phi'(a)) \rangle = 0. \]

Observe that $(1, \phi'(a))$ is the tangent line to the implicitly defined function at $x = a$, since in a neighborhood of $(a,b)$ the graph of that function is the curve $x \mapsto (x, \phi(x))$.

69. Suppose the function $\phi : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and that $\phi'(a) \neq 0$. Set $b = \phi(a)$ and define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = y - \phi(x)$ for $(x,y) \in \mathbb{R}^2$. Apply Dini’s theorem to the function $f : \mathbb{R}^2 \to \mathbb{R}$ at the point $(a,b)$ and compare the result with the conclusion of the Inverse Function Theorem applied to $\phi$ at $a$.

*Solution.* Since $D_1 f(a,b) = -\phi'(a) \neq 0$, by Dini’s theorem, there is a continuously differentiable function $\psi$ defined on a neighborhood of $b$ such that $x = \psi(y)$ uniquely solves the equation $f(x,y) = y - \phi(x) = 0$ in a neighborhood of $(a,b)$. Thus, for $y$ in a neighborhood of $b$, $\psi(y) = \phi^{-1}(y)$ and

\[ 0 = D_1 f(\psi(y), y) \psi'(y) + D_2 f(\psi(y), y) = -\phi'(\phi^{-1}(y))(\phi^{-1})'(y) + 1. \quad (9) \]
Since $\phi'(a) \neq 0$, the Inverse Function Theorem shows that in a neighborhood of $b$, $\phi$ has a continuously differentiable inverse and for $y$ in this neighborhood,

$$ (\phi^{-1})'(y) = \phi'(\phi^{-1}(y))^{-1}. $$

(10)

Observe that equations (9) and (10) are simply rearrangements of one another.

70. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable and $c > 0$ is such that

$$ D_2 f(x, y) \geq c \quad \text{for all } (x, y) \in \mathbb{R}^2. $$

(11)

Prove there is a continuously differentiable function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, g(x)) = 0$ for all $x \in \mathbb{R}$ and if $f(x, y) = 0$, then $y = g(x)$.

Solution. Fix $x$ and define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(y) = f(x, y)$. The assumption (11) implies $\phi'(y) \geq c > 0$. Thus, $\phi$ has a unique root (why?), call it $g(x)$. That is,

$$ \phi(g(x)) = f(x, g(x)) = 0. $$

Uniqueness of the root (for arbitrary $x$) shows that if $f(x, y) = 0$ then $y = g(x)$. To see that $g$ is continuously differentiable at an arbitrary point $a \in \mathbb{R}$, note that by (11), Dini’s theorem holds at the point $(a, g(a))$. Since $g$ is unique, it must agree with the implicit function furnished by Dini’s theorem in a neighborhood of $a$. Since the latter function is continuously differentiable at $a$, so is $g$.

71. Consider the linear system of equations

$$ a_{11}x + a_{12}y + a_{13}z = 0, $$

$$ a_{21}x + a_{22}y + a_{23}z = 0, $$

$(x, y, z) \in \mathbb{R}^3$.

(12)

Define $\nu = (a_{11}, a_{12}, a_{13})$ and $\beta = (a_{21}, a_{22}, a_{23})$. Show that if $\nu \times \beta \neq (0, 0, 0)$, then the equations (12) defines two of the variables as a function of the remaining variable. Interpret this geometrically.

Solution. Define

$$ f(x, y, z) = (a_{11}x + a_{12}y + a_{13}z, a_{21}x + a_{22}y + a_{23}z) $$

and observe that

$$ \nu \times \beta = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} (1, 0, 0) - \begin{vmatrix} a_{11} & a_{13} \\ a_{12} & a_{23} \end{vmatrix} (0, 1, 0) + \begin{vmatrix} a_{12} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} (0, 0, 1) $$

$$ = |D_{(y,z)}f(x, y, z)|(1, 0, 0) - |D_{(x,z)}f(x, y, z)|(0, 1, 0) + |D_{(x,y)}f(x, y, z)|(0, 0, 1). $$

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Since $\nu \times \beta \neq (0, 0, 0)$, one of the determinants above must be nonzero. Thus, the Implicit Function Theorem shows that (12) implicitly defines two of the variables as a function of the third. Geometrically, each equation of (12) represents a plane in $\mathbb{R}^3$, and the assumption $\nu \times \beta \neq (0, 0, 0)$ guarantees that these two planes are not parallel. Thus, the planes intersect in a line, which is why we can solve for two of the variables as a function of the third.

72. Consider the solutions of the equation $y^3 - x^2 = 0$ for $(x, y) \in \mathbb{R}^2$. Does the Implicit Function Theorem apply at the point $(0, 0)$? Does this equation define one of the components of a solution $(x, y)$ as a function of the other component?

Solution. The hypotheses of the Implicit Function Theorem do not hold at $(0, 0)$: letting $f(x, y) = y^3 - x^2$, we have $D_1f(0, 0) = D_2f(0, 0) = 0$. However, the equation $f(x, y) = 0$ does define $y$ as a function of $x$, namely $y = x^{2/3}$. (Note, however, that $y$ is not a continuously differentiable function of $x$.)