369 Midterm 3

Your Name:

Instructions: This test contains 5 questions. Each question is worth 20 points. Read each question carefully before responding. ***Give justifications to all of your responses, unless otherwise specified.*** No calculators are allowed. One standard sized page of notes with writing on one side is allowed.

1. Find the QR-factorization of

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution. The factorization is

$$A = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} & 0 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{5} & 4/\sqrt{5} & 0 \\ 0 & 3/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2. Let

$$A = \begin{pmatrix} 2/3 & 0 & 4 \\ 1/2 & 1/\sqrt{2} & 5 \\ -1/2 & 1/\sqrt{2} & 6 \end{pmatrix}.$$

and let $A = QR$ be the QR-factorization of A. What is the $(2,3)$-entry of R?

Note: you do NOT have to compute the whole QR-factorization.

Solution. The $(2,3)$ entry of R is $u_2 \cdot v_3$, where u_1, u_2, u_3 are the columns of Q in the QR-factorization. Note that u_2 is equal to the second column of A, since it is a unit vector orthogonal to the first column. Thus,

$$(2,3) \text{ entry of } R = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = 11/\sqrt{2}.$$

3. Use Gram-Schmidt to find a vector orthogonal to $\operatorname{span}\{v_1, v_2\}$, where

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

Solution. Using Gram-Schmidt on v_1, v_2 and $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (which is not in $\operatorname{span}\{v_1, v_2\}$), we find

$$w_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 1/3 \\ -1/3 \\ 2/3 \end{pmatrix},$$

and w_3 is orthogonal to $\operatorname{span}\{v_1, v_2\}$.

4. Find $x \in \mathbb{R}^2$ that minimizes $|Ax - b|$, where

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Solution. The solution x satisfies $Rx = Q^t b$ where

$$Q = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \end{pmatrix}$$

is obtained by normalizing w_1, w_2 from Problem 3, and

$$R = Q^t A = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{pmatrix}.$$

From this we get $x = \begin{pmatrix} 5/6 \\ 2/3 \end{pmatrix}$.

5. Let $u, v \in \mathbb{R}^n$ be orthogonal unit vectors, and let $w \in \mathbb{R}^n$ be another vector. Prove that $w - \text{proj}_u w - \text{proj}_v w$ is orthogonal to $\text{span}\{u, v\}$ (or equal to 0).

Solution. Since u, v are unit vectors, $w - \text{proj}_u w - \text{proj}_v w = w - (w \cdot u)u - (w \cdot v)v$. Since

$$(w - (w \cdot u)u - (w \cdot v)v) \cdot u = w \cdot u - (w \cdot u)(u \cdot u) - (w \cdot v)(v \cdot u) = w \cdot u - w \cdot u = 0$$
$$(w - (w \cdot u)v - (w \cdot v)v) \cdot v = w \cdot v - (w \cdot u)(u \cdot v) - (w \cdot v)(v \cdot v) = w \cdot v - w \cdot v = 0,$$

$w - \text{proj}_u w - \text{proj}_v w$ is (zero or) orthogonal to both u and v, hence orthogonal to $\text{span}\{u, v\}$.