Thus, $\lambda^{(4)} = 4.99999$ is the first approximation whose estimated percentage error is less than 0.1%.

Remark A rule for deciding when to stop an iterative process is called a ***stopping procedure***. In the exercises, we will discuss stopping procedures for the power method that are based on the dominant eigenvector rather than the dominant eigenvalue.

Concept Review

- Power sequence
- Dominant eigenvalue
- Dominant eigenvector
- Power method with Euclidean scaling
- Rayleigh quotient
- Power method with maximum entry scaling
- Relative error
- Percentage error
- Estimated relative error
- Estimated percentage error
- Stopping procedure

Skills

- Identify the dominant eigenvalue of a matrix.
- Use the power methods described in this section to approximate a dominant eigenvector.
- Find the estimated relative and percentage errors associated with the power methods.

Exercise Set 9.2

In Exercises 1–2, the distinct eigenvalues of a matrix are given. Determine whether A has a dominant eigenvalue, and if so, find it.

1. (a) $\lambda_1 = 7,\ \lambda_2 = 3,\ \lambda_3 = -8,\ \lambda_4 = 1$

(b) $\lambda_1 = -5,\ \lambda_2 = 3,\ \lambda_3 = 2,\ \lambda_4 = 5$

Answer:

(a) λ_3 dominant

(b) No dominant eigenvalue

2. (a) $\lambda_1 = 1,\ \lambda_2 = 0,\ \lambda_3 = -3,\ \lambda_4 = 2$

(b) $\lambda_1 = -3,\ \lambda_2 = -2,\ \lambda_3 = -1,\ \lambda_4 = 3$

In Exercises 3–4, apply the power method with Euclidean scaling to the matrix A, starting with $\mathbf{x}_0$ and stopping at $\mathbf{x}_4$. Compare the resulting approximations to the exact values of the dominant eigenvalue and the corresponding unit eigenvector.

3. $A=\begin{bmatrix} 5 & -1 \\ -1 & -1 \end{bmatrix}$; $\quad \mathbf{x}_0=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Answer:

$\mathbf{x}_1 \approx \begin{bmatrix} 0.98058 \\ -0.19612 \end{bmatrix}$; $\mathbf{x}_2 \approx \begin{bmatrix} 0.98837 \\ -0.15206 \end{bmatrix}$; $\mathbf{x}_3 \approx \begin{bmatrix} 0.98679 \\ -0.16201 \end{bmatrix}$; $\mathbf{x}_4 \approx \begin{bmatrix} 0.98715 \\ -0.15977 \end{bmatrix}$;

dominant eigenvalue: $\lambda = 2+\sqrt{10} \approx 5.16228$;

dominant eigenvector: $\begin{bmatrix} 1 \\ 3-\sqrt{10} \end{bmatrix} \approx \begin{bmatrix} 1 \\ -0.16228 \end{bmatrix}$

4. $A=\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$; $\quad \mathbf{x}_0=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

In Exercises 5–6, apply the power method with maximum entry scaling to the matrix A, starting with $\mathbf{x}_0$ and stopping at $\mathbf{x}_4$. Compare the resulting approximations to the exact values of the dominant eigenvalue and the corresponding scaled eigenvector.

5. $A=\begin{bmatrix} 1 & -3 \\ -3 & 5 \end{bmatrix}$; $\quad \mathbf{x}_0=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Answer:

$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\lambda^{(1)}=6$; $\mathbf{x}_2 = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}$, $\lambda^{(2)}=6.6$; $\mathbf{x}_3 \approx \begin{bmatrix} -0.53846 \\ 1 \end{bmatrix}$, $\lambda^{(3)} \approx 6.60550$;

$\mathbf{x}_4 \approx \begin{bmatrix} -0.53488 \\ 1 \end{bmatrix}$, $\lambda^{(4)} \approx 6.60555$;

dominant eigenvalue: $\lambda = 3+\sqrt{13} \approx 6.60555$;

dominant eigenvector: $\begin{bmatrix} -\dfrac{3}{\sqrt{26+4\sqrt{13}}} \\ \dfrac{2+\sqrt{13}}{\sqrt{26+4\sqrt{13}}} \end{bmatrix} \approx \begin{bmatrix} -0.47186 \\ 0.88167 \end{bmatrix}$

6. $A=\begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}$; $\quad \mathbf{x}_0=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

7. Let

$$A=\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}; \quad \mathbf{x}_0=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(a) Use the power method with maximum entry scaling to approximate a dominant eigenvector of A. Start with $\mathbf{x}_0$, round off all computations to three decimal places, and stop after three iterations.

(b) Use the result in part (a) and the Rayleigh quotient to approximate the dominant eigenvalue of A.

(c) Find the exact values of the eigenvector and eigenvalue approximated in parts (a) and (b).

(d) Find the percentage error in the approximation of the dominant eigenvalue.

Answer:

(a) $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -0.8 \end{bmatrix}$, $\mathbf{x}_3 \approx \begin{bmatrix} 1 \\ -0.929 \end{bmatrix}$

(b) $\lambda^{(1)} = 2.8,\ \lambda^{(2)} \approx 2.976,\ \lambda^{(3)} \approx 2.997$

(c) Dominant eigenvalue: $\lambda = 3$; dominant eigenvector: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(d) 0.1%

8. Repeat the directions of Exercise 7 with

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 10 \end{bmatrix}; \quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

In Exercises 9–10, a matrix A with a dominant eigenvalue and a sequence $\mathbf{x}_0, A\mathbf{x}_0, \ldots, A^5\mathbf{x}_0$ are given. Use Formulas 9 and 10 to approximate the dominant eigenvalue and a corresponding eigenvector.

9. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$; $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A\mathbf{x}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $A^2\mathbf{x}_0 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$,

$A^3\mathbf{x}_0 = \begin{bmatrix} 13 \\ 14 \end{bmatrix}$, $A^4\mathbf{x}_0 = \begin{bmatrix} 41 \\ 40 \end{bmatrix}$, $A^5\mathbf{x}_0 = \begin{bmatrix} 121 \\ 122 \end{bmatrix}$

Answer:

$2.99993;\ \begin{bmatrix} 0.99180 \\ 1.00000 \end{bmatrix}$

10. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$; $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $A\mathbf{x}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $A^2\mathbf{x}_0 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$,

$A^3\mathbf{x}_0 = \begin{bmatrix} 14 \\ 13 \end{bmatrix}$, $A^4\mathbf{x}_0 = \begin{bmatrix} 40 \\ 41 \end{bmatrix}$, $A^5\mathbf{x}_0 = \begin{bmatrix} 122 \\ 121 \end{bmatrix}$

11. Consider matrices

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_0 = \begin{bmatrix} a \\ b \end{bmatrix}$$

where $\mathbf{x}_0$ is a unit vector and $a \neq 0$. Show that even though the matrix A is symmetric and has a dominant eigenvalue, the power sequence 1 in Theorem 9.2.1 does not converge. This shows that the requirement in that theorem that the dominant eigenvalue be positive is essential.

12. Use the power method with Euclidean scaling to approximate the dominant eigenvalue and a corresponding eigenvector of A. Choose your own starting vector, and stop when the estimated percentage error in the eigenvalue approximation is less than 0.1%.

(a) $\begin{bmatrix} 1 & 3 & 3 \\ 3 & 4 & -1 \\ 3 & -1 & 10 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & 1 \\ 1 & -1 & 4 & 1 \\ 1 & 1 & 1 & 8 \end{bmatrix}$

13. Repeat Exercise 12, but this time stop when all corresponding entries in two successive eigenvector approximations differ by less than 0.01 in absolute value.

Answer:

(a) Starting with $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, it takes 8 iterations.

(b) Starting with $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, it takes 8 iterations.

14. Repeat Exercise 12 using maximum entry scaling.

15. Prove: If A is a nonzero $n \times n$ matrix, then $A^T A$ and AA^T have positive dominant eigenvalues.

16. ***(For readers familiar with proof by induction***) Let A be an $n \times n$ matrix, let $\mathbf{x}_0$ be a unit vector in R^n, and define the sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k, \ldots$ by

$$\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\|A\mathbf{x}_0\|}, \quad \mathbf{x}_2 = \frac{A\mathbf{x}_1}{\|A\mathbf{x}_1\|}, \ldots, \quad \mathbf{x}_k = \frac{A\mathbf{x}_{k-1}}{\|A\mathbf{x}_{k-1}\|}, \ldots$$

Prove by induction that $\mathbf{x}_k = A^k\mathbf{x}_0 \,/\, \|A^k\mathbf{x}_0\|$.

17. ***(For readers familiar with proof by induction)*** Let A be an $n \times n$ matrix, let $\mathbf{x}_0$ be a nonzero vector in R^n, and define the sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k, \ldots$ by

$$\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\max(A\mathbf{x}_0)}, \quad \mathbf{x}_2 = \frac{A\mathbf{x}_1}{\max(A\mathbf{x}_1)}, \quad \cdots, \quad \mathbf{x}_k = \frac{A\mathbf{x}_{k-1}}{\max(A\mathbf{x}_{k-1})}, \quad \cdots$$

Prove by induction that

$$\mathbf{x}_k = \frac{A^k\mathbf{x}_0}{\max\left(A^k\mathbf{x}_0\right)}$$