Concept Review

- Rank
- Nullity
- Dimension Theorem
- Overdetermined system
- Underdetermined system
- Fundamental spaces of a matrix
- Relationships among the fundamental spaces
- Orthogonal complement
- Equivalent characterizations of invertible matrices

Skills

- Find the rank and nullity of a matrix.
- Find the dimension of the row space of a matrix.

Exercise Set 4.8

1. Verify that $\operatorname{rank}(A) = \operatorname{rank}(A^T)$.

$$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix}$$

Answer:

$\operatorname{Rank}(A) = \operatorname{Rank}(A^T) = 2$

2. Find the rank and nullity of the matrix; then verify that the values obtained satisfy Formula 4 in the Dimension Theorem.

(a) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

(e) $A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$

3. In each part of Exercise 2, use the results obtained to find the number of leading variables and the number of parameters in the solution of $A\mathbf{x} = \mathbf{0}$ without solving the system.

Answer:

(a) 2; 1
(b) 1; 2
(c) 2; 2
(d) 2; 3
(e) 3; 2

4. In each part, use the information in the table to find the dimension of the row space of *A*, column space of *A*, null space of *A*, and null space of A^T.

	(a)	**(b)**	**(c)**	**(d)**	**(e)**	**(f)**	**(g)**
Size of *A*	3×3	3×3	3×3	5×9	9×5	4×4	6×2
Rank(*A*)	3	2	1	2	2	0	2

5. In each part, find the largest possible value for the rank of *A* and the smallest possible value for the nullity of *A*.

(a) *A* is 4×4
(b) *A* is 3×5
(c) *A* is 5×3

Answer:

(a) $\text{Rank} = 4,\ \text{nullity} = 0$
(b) $\text{Rank} = 3,\ \text{nullity} = 2$
(c) $\text{Rank} = 3,\ \text{nullity} = 0$

6. If *A* is an $m \times n$ matrix, what is the largest possible value for its rank and the smallest possible value for its nullity?

7. In each part, use the information in the table to determine whether the linear system $A\mathbf{x} = \mathbf{b}$ is consistent. If so, state the number of parameters in its general solution.

	(a)	**(b)**	**(c)**	**(d)**	**(e)**	**(f)**	**(g)**
Size of *A*	3×3	3×3	3×3	5×9	5×9	4×4	6×2
Rank (*A*)	3	2	1	2	2	0	2
Rank [A |**b**]	3	3	1	2	3	0	2

Answer:

(a) Yes, 0

(b) No

(c) Yes, 2

(d) Yes, 7

(e) No

(f) Yes, 4

(g) Yes, 0

8. For each of the matrices in Exercise 7, find the nullity of A, and determine the number of parameters in the general solution of the homogeneous linear system $A\mathbf{x}=0$.

9. What conditions must be satisfied by b_1, b_2, b_3, b_4, and b_5 for the overdetermined linear system

$$\begin{aligned} x_1-3x_2&=b_1\\ x_1-2x_2&=b_2\\ x_1+x_2&=b_3\\ x_1-4x_2&=b_4\\ x_1+5x_2&=b_5 \end{aligned}$$

to be consistent?

Answer:

$b_1=r,\ b_2=s,\ b_3=4s-3r,\ b_4=2r-s,\ b_5=8s-7r$

10. Let

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Show that A has $\text{rank } 2$ if and only if one or more of the determinants

$$\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22} \end{vmatrix},\quad \begin{vmatrix} a_{11} & a_{13}\\ a_{21} & a_{23} \end{vmatrix},\quad \begin{vmatrix} a_{12} & a_{13}\\ a_{22} & a_{23} \end{vmatrix}$$

is nonzero.

11. Suppose that A is a 3×3 matrix whose null space is a line through the origin in 3-space. Can the row or column space of A also be a line through the origin? Explain.

Answer:

No

12. Discuss how the rank of A varies with t.

(a) $$A=\begin{bmatrix} 1 & 1 & t\\ 1 & t & 1\\ t & 1 & 1 \end{bmatrix}$$

(b) $$A=\begin{bmatrix} t & 3 & -1\\ 3 & 6 & -2\\ -1 & -3 & t \end{bmatrix}$$

13. Are there values of r and s for which

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

has rank 1? Has rank 2? If so, find those values.

Answer:

Rank is 2 if $r=2$ and $s=1$; the rank is never 1.

14. Use the result in Exercise 10 to show that the set of points (x, y, z) in R^3 for which the matrix

$$\begin{bmatrix} x & y & z \\ 1 & x & y \end{bmatrix}$$

has $\operatorname{rank} 1$ is the curve with parametric equations $x=t$, $y=t^2$, $z=t^3$.

15. Prove: If $k \neq 0$, then A and kA have the same rank.

16. (a) Give an example of a 3×3 matrix whose column space is a plane through the origin in 3-space.

(b) What kind of geometric object is the null space of your matrix?

(c) What kind of geometric object is the row space of your matrix?

17. (a) If A is a 3×5 matrix, then the number of leading 1's in the reduced row echelon form of A is at most __________ . Why?

(b) If A is a 3×5 matrix, then the number of parameters in the general solution of $A\mathbf{x}=0$ is at most __________ . Why?

(c) If A is a 5×3 matrix, then the number of leading 1's in the reduced row echelon form of A is at most __________ . Why?

(d) If A is a 5×3 matrix, then the number of parameters in the general solution of $A\mathbf{x}=0$ is at most __________ . Why?

Answer:

(a) 3

(b) 5

(c) 3

(d) 3

18. (a) If A is a 3×5 matrix, then the rank of A is at most __________ . Why?

(b) If A is a 3×5 matrix, then the nullity of A is at most __________ . Why?

(c) If A is a 3×5 matrix, then the rank of A^T is at most __________ . Why?

(d) If A is a 3×5 matrix, then the nullity of A^T is at most __________ . Why?

19. Find matrices A and B for which $\operatorname{rank}(A)=\operatorname{rank}(B)$, but $\operatorname{rank}\left(A^2\right) \neq \operatorname{rank}\left(B^2\right)$.

Answer:

$$A=\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix};\ B=\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$$

20. Prove: If a matrix A is not square, then either the row vectors or the column vectors of A are linearly dependent.

True-False Exercises

In parts (a)–(j) determine whether the statement is true or false, and justify your answer.

(a) Either the row vectors or the column vectors of a square matrix are linearly independent.

Answer:

False

(b) A matrix with linearly independent row vectors and linearly independent column vectors is square.

Answer:

True

(c) The nullity of a nonzero $m \times n$ matrix is at most m.

Answer:

False

(d) Adding one additional column to a matrix increases its rank by one.

Answer:

False

(e) The nullity of a square matrix with linearly dependent rows is at least one.

Answer:

True

(f) If A is square and $A\mathbf{x}=\mathbf{b}$ is inconsistent for some vector $\mathbf{b}$, then the nullity of A is zero.

Answer:

False

(g) If a matrix A has more rows than columns, then the dimension of the row space is greater than the dimension of the column space.

Answer:

False

(h) If $\operatorname{rank}(A^T)=\operatorname{rank}(A)$, then A is square.

Answer:

False

(i) There is no 3×3 matrix whose row space and null space are both lines in 3-space.

Answer:

True

(j) If V is a subspace of R^n and W is a subspace of V, then $W^\perp$ is a subspace of $V^\perp$.

Answer:

False