$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_{r-1}\mathbf{v}_{r-1} + k_r(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{r-1}\mathbf{v}_{r-1})$$

which expresses **w** as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{r-1}$.

Proof of Theorem 4.5.5(a) If S is a set of vectors that spans V but is not a basis for V, then S is a linearly dependent set. Thus some vector **v** in S is expressible as a linear combination of the other vectors in S. By the Plus/Minus Theorem (4.5.3*b*), we can remove **v** from S, and the resulting set S' will still span V. If S' is linearly independent, then S' is a basis for V, and we are done. If S' is linearly dependent, then we can remove some appropriate vector from S' to produce a set S'' that still spans V. We can continue removing vectors in this way until we finally arrive at a set of vectors in S that is linearly independent and spans V. This subset of S is a basis for V.

Proof of Theorem 4.5.5(b) Suppose that $\dim(V) = n$. If S is a linearly independent set that is not already a basis for V, then S fails to span V, so there is some vector **v** in V that is not in $\text{span}(S)$. By the Plus/Minus Theorem (4.5.3*a*), we can insert v into S, and the resulting set S' will still be linearly independent. If S' spans V, then S' is a basis for V, and we are finished. If S' does not span V, then we can insert an appropriate vector into S' to produce a set S'' that is still linearly independent. We can continue inserting vectors in this way until we reach a set with n linearly independent vectors in V. This set will be a basis for V by Theorem 4.5.4.

Concept Review

- Dimension
- Relationships among the concepts of linear independence, basis, and dimension

Skills

- Find a basis for and the dimension of the solution space of a homogeneous linear system.
- Use dimension to determine whether a set of vectors is a basis for a finite-dimensional vector space.
- Extend a linearly independent set to a basis.

Exercise Set 4.5

In Exercises 1–6, find a basis for the solution space of the homogeneous linear system, and find the dimension of that space.

1.
$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ -2x_1 - x_2 + 2x_3 &= 0 \\ -x_1 + x_3 &= 0 \end{aligned}$$

Answer:

Basis: (1, 0, 1); dimension = 1

2. $3x_1 + x_2 + x_3 + x_4 = 0$
$5x_1 - x_2 + x_3 - x_4 = 0$

3. $x_1 - 4x_2 + 3x_3 - x_4 = 0$
$2x_1 - 8x_2 + 6x_3 - 2x_4 = 0$

Answer:

Basis: $(4, 1, 0, 0), \ (-3, 0, 1, 0), \ (1, 0, 0, 1)$; $\text{dimension} = 3$

4. $x_1 - 3x_2 + x_3 = 0$
$2x_1 - 6x_2 + 2x_3 = 0$
$3x_1 - 9x_2 + 3x_3 = 0$

5. $2x_1 + x_2 + 3x_3 = 0$
$x_1 + 5x_3 = 0$
$x_2 + x_3 = 0$

Answer:

No basis; $\text{dimension} = 0$

6. $x + y + z = 0$
$3x + 2y - 2z = 0$
$4x + 3y - z = 0$
$6x + 5y + z = 0$

7. Find bases for the following subspaces of R^3.

(a) The plane $3x - 2y + 5z = 0$.
(b) The plane $x - y = 0$.
(c) The line $x = 2t, y = -t, z = 4t$.
(d) All vectors of the form (a, b, c), where $b = a + c$.

Answer:

(a) $\left(\frac{2}{3}, 1, 0\right), \ \left(-\frac{5}{3}, 0, 1\right)$
(b) (1, 1, 0), (0, 0, 1)
(c) $(2, -1, 4)$
(d) (1, 1, 0), (0, 1, 1)

8. Find the dimensions of the following subspaces of R^4.

(a) All vectors of the form $(a, b, c, 0)$.
(b) All vectors of the form (a, b, c, d), where $d = a + b$ and $c = a - b$.
(c) All vectors ofthe form (a, b, c, d), where $a = b = c = d$.

9. Find the dimension of each ofthe following vector spaces.

(a) The vector space of all diagonal $n \times n$ matrices.

(b) The vector space of all symmetric $n \times n$ matrices.

(c) The vector space of all upper triangular $n \times n$ matrices.

Answer:

(a) n

(b) $\frac{n(n+1)}{2}$

(c) $\frac{n(n+1)}{2}$

10. Find the dimension of the subspace of P_3 consisting of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$.

11. (a) Show that the set W of all polynomials in P_2 such that $p(1) = 0$ is a subspace of P_2.

(b) Make a conjecture about the dimension of W.

(c) Confirm your conjecture by finding a basis for W.

12. Find a standard basis vector for R^3 that can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for R^3.

(a) $\mathbf{v}_1 = (-1, 2, 3),\ \mathbf{v}_2 = (1, -2, -2)$

(b) $\mathbf{v}_1 = (1, -1, 0),\ \mathbf{v}_2 = (3, 1, -2)$

13. Find standard basis vectors for R^4 that can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for R^4.

$$\mathbf{v}_1 = (1, -4, 2, -3),\ \mathbf{v}_2 = (-3, 8, -4, 6)$$

Answer:

Any two of (0, 1, 0, 0), (0, 0, 1, 0), and (0, 0, 0, 1) can be used.

14. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for a vector space V. Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is also a basis, where $\mathbf{u}_1 = \mathbf{v}_1$, $\mathbf{u}_2 = \mathbf{v}_1 + \mathbf{v}_2$, and $\mathbf{u}_3 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$.

15. The vectors $\mathbf{v}_1 = (1, -2, 3)$ and $\mathbf{v}_2 = (0, 5, -3)$ are linearly independent. Enlarge $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for R^3.

Answer:

$\mathbf{v}_3 = (a, b, c)$ with $9a - 3b - 5c \neq 0$

16. The vectors $\mathbf{v}_1 = (1, -2, 3, -5)$ and $\mathbf{v}_2 = (0, -1, 2, -3)$ are linearly independent. Enlarge $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for R^4.

17. (a) Show that for every positive integer n, one can find $n + 1$ linearly independent vectors in $F(-\infty, \infty)$. [***Hint:*** Look for polynomials.]

(b) Use the result inpart (a) to prove that $F(-\infty, \infty)$ is infinite- dimensional.

(c) Prove that $C(-\infty, \infty)$, $C^m(-\infty, \infty)$, and $C^\infty(-\infty, \infty)$ are infinite-dimensional vector spaces.

18. Let S be a basis for an n-dimensional vector space V. Show that if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ form a linearly independent set of vectors in V, then the coordinate vectors $(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, \ldots, (\mathbf{v}_r)_S$ form a linearly independent set in R^n, and conversely.

19. Using the notation from Exercise 18, show that if the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ span V, then the coordinate vectors $(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, \ldots, (\mathbf{v}_r)_S$ span R^n, and conversely.

20. Find a basis for the subspace of P_2 spanned by the given vectors.

(a) $-1+x-2x^2, 3+3x+6x^2, 9$

(b) $1+x, x^2, -2+2x^2, -3x$

(c) $1+x-3x^2, 2+2x-6x^2, 3+3x-9x^2$

[*Hint:* Let S be the standard basis for P_2, and work with the coordinate vectors relative to S as in Exercises 18 and 19.]

21. Prove: A subspace of a finite-dimensional vector space is finite-dimensional.

22. State the two parts of Theorem 4.5.2 in contrapositive form.

True-False Exercises

In parts (a)–(j) determine whether the statement is true or false, and justify your answer.

(a) The zero vector space has dimension zero.

Answer:

True

(b) There is a set of 17 linearly independent vectors in R^{17}.

Answer:

True

(c) There is a set of 11 vectors that span R^{17}.

Answer:

False

(d) Every linearly independent set of five vectors in R^5 is a basis for R^5.

Answer:

True

(e) Every set of five vectors that spans R^5 is a basis for R^5.

Answer:

True

(f) Every set of vectors that spans R^n contains a basis for R^n.

Answer:

True

(g) Every linearly independent set of vectors in R^n is contained in some basis for R^n.

Answer:

True

(h) There is a basis for M_{22} consisting of invertible matrices.

Answer:

True

(i) If A has size $n \times n$ and $I_n, A, A^2, \ldots, A^{n^2}$ are distinct matrices, then $\left\{I_n, A, A^2, \ldots, A^{n^2}\right\}$ is linearly dependent.

Answer:

True

(j) There are at least two distinct three-dimensional subspaces of P_2.

Answer:

False