$$\mathbf{v}_1 - c_2\mathbf{v}_2 - c_3\mathbf{v}_3 - \cdots - c_r\mathbf{v}_r = 0$$

It follows that S is linearly dependent since the equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = 0$$

is satisfied by

$$k_1 = 1, \quad k_2 = -c_2, \ldots, \quad k_r = -c_r$$

which are not all zero. The proof in the case where some vector other than $\mathbf{v}_1$ is expressible as a linear combination of the other vectors in S is similar.

Concept Review

- Trivial solution
- Linearly independent set
- Linearly dependent set
- Wronskian

Skills

- Determine whether a set of vectors is linearly independent or linearly dependent.
- Express one vector in a linearly dependent set as a linear combination of the other vectors in the set.
- Use the Wronskian to show that a set of functions is linearly independent.

Exercise Set 4.3

1. Explain why the following are linearly dependent sets of vectors. (Solve this problem by inspection.)

(a) $\mathbf{u}_1 = (-1, 2, 4)$ and $\mathbf{u}_2 = (5, -10, -20)$ in R^3

(b) $\mathbf{u}_1 = (3, -1)$, $\mathbf{u}_2 = (4, 5)$, $\mathbf{u}_3 = (-4, 7)$ in R^2

(c) $\mathbf{p}_1 = 3 - 2x + x^2$ and $\mathbf{p}_2 = 6 - 4x + 2x^2$ in P_2

(d) $A = \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -4 \\ -2 & 0 \end{bmatrix}$ in M_{22}

Answer:

(a) $\mathbf{u}_2$ is a scalar multiple of $\mathbf{u}_1$.

(b) The vectors are linearly dependent by Theorem 4.3.3.

(c) $\mathbf{p}_2$ is a scalar multiple of $\mathbf{p}_1$.

(d) B is a scalar multiple of A.

2. Which of the following sets of vectors in R^3 are linearly dependent?

(a) $(4, -1, 2)$, $(-4, 10, 2)$

(b) $(-3, 0, 4)$, $(5, -1, 2)$, $(1, 1, 3)$

(c) $(8, -1, 3)$, $(4, 0, 1)$

(d) $(-2, 0, 1)$, $(3, 2, 5)$, $(6, -1, 1)$, $(7, 0, -2)$

3. Which of the following sets of vectors in R^4 are linearly dependent?

(a) $(3, 8, 7, -3), (1, 5, 3, -1), (2, -1, 2, 6), (1, 4, 0, 3)$

(b) $(0, 0, 2, 2), (3, 3, 0, 0), (1, 1, 0, -1)$

(c) $(0, 3, -3, -6), (-2, 0, 0, -6), (0, -4, -2, -2), (0, -8, 4, -4)$

(d) $(3, 0, -3, 6), (0, 2, 3, 1), (0, -2, -2, 0), (-2, 1, 2, 1)$

Answer:

None

4. Which of the following sets of vectors in P_2 are linearly dependent?

(a) $2 - x + 4x^2, 3 + 6x + 2x^2, 2 + 10x - 4x^2$

(b) $3 + x + x^2, 2 - x + 5x^2, 4 - 3x^2$

(c) $6 - x^2$

(d) $1 + 3x + 3x^2, x + 4x^2, 5 + 6x + 3x^2, 7 + 2x - x^2$

5. Assume that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are vectors in R^3 that have their initial points at the origin. In each part, determine whether the three vectors lie in a plane.

(a) $\mathbf{v}_1 = (2, -2, 0), \mathbf{v}_2 = (6, 1, 4), \mathbf{v}_3 = (2, 0, -4)$

(b) $\mathbf{v}_1 = (-6, 7, 2), \mathbf{v}_2 = (3, 2, 4), \mathbf{v}_3 = (4, -1, 2)$

Answer:

(a) They do not lie in a plane.

(b) They do lie in a plane.

6. Assume that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are vectors in R^3 that have their initial points at the origin. In each part, determine whether the three vectors lie on the same line.

(a) $\mathbf{v}_1 = (-1, 2, 3), \mathbf{v}_2 = (2, -4, -6), \mathbf{v}_3 = (-3, 6, 0)$

(b) $\mathbf{v}_1 = (2, -1, 4), \mathbf{v}_2 = (4, 2, 3), \mathbf{v}_3 = (2, 7, -6)$

(c) $\mathbf{v}_1 = (4, 6, 8), \mathbf{v}_2 = (2, 3, 4), \mathbf{v}_3 = (-2, -3, -4)$

7. (a) Show that the three vectors $\mathbf{v}_1 = (0, 3, 1, -1), \mathbf{v}_2 = (6, 0, 5, 1)$, and $\mathbf{v}_3 = (4, -7, 1, 3)$ form a linearly dependent set in R^4.

(b) Express each vector in part (a) as a linear combination of the other two.

Answer:

(b) $\mathbf{v}_1 = \frac{2}{7}\mathbf{v}_2 - \frac{3}{7}\mathbf{v}_3,\ \mathbf{v}_2 = \frac{7}{2}\mathbf{v}_1 + \frac{3}{2}\mathbf{v}_3,\ \mathbf{v}_3 = -\frac{7}{3}\mathbf{v}_1 + \frac{2}{3}\mathbf{v}_2$

8. (a) Show that the three vectors $\mathbf{v}_1 = (1, 2, 3, 4)$, $\mathbf{v}_2 = (0, 1, 0, -1)$, and $\mathbf{v}_3 = (1, 3, 3, 3)$ form a linearly dependent set in R^4.

(b) Express each vector in part (a) as a linear combination of the other two.

9. For which real values of λ do the following vectors form a linearly dependent set in R^3?

$$\mathbf{v}_1 = \left(\lambda, -\frac{1}{2}, -\frac{1}{2}\right), \quad \mathbf{v}_2 = \left(-\frac{1}{2}, \lambda, -\frac{1}{2}\right), \quad \mathbf{v}_3 = \left(-\frac{1}{2}, -\frac{1}{2}, \lambda\right)$$

Answer:

$\lambda = -\frac{1}{2}$, $\lambda = 1$

10. Show that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of vectors, then so are $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, $\{\mathbf{v}_2, \mathbf{v}_3\}$, $\{\mathbf{v}_1\}$, $\{\mathbf{v}_2\}$, and $\{\mathbf{v}_3\}$.

11. Show that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a linearly independent set of vectors, then so is every nonempty subset of S.

12. Show that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set of vectors in a vector space V, and $\mathbf{v}_4$ is any vector in V that is not in S, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is also linearly dependent.

13. Show that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a linearly dependent set of vectors in a vector space V, and if $\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n$ are any vectors in V that are not in S, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$ is also linearly dependent.

14. Show that in P_2 every set with more than three vectors is linearly dependent.

15. Show that if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and $\mathbf{v}_3$ does not lie in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

16. Prove: For any vectors **u**, **v**, and **w** in a vector space V, the vectors $\mathbf{u} - \mathbf{v}$, $\mathbf{v} - \mathbf{w}$, and $\mathbf{w} - \mathbf{u}$ form a linearly dependent set.

17. Prove: The space spanned by two vectors in R^3 is a line through the origin, a plane through the origin, or the origin itself.

18. Under what conditions is a set with one vector linearly independent?

19. Are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in part (*a*) of the accompanying figure linearly independent? What about those in part (*b*)? Explain.

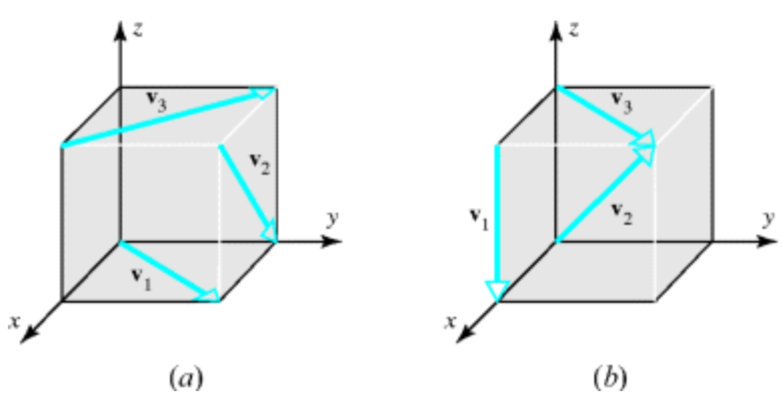

Figure Ex-19

Answer:

(a) They are linearly independent since $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ do not lie in the same plane when they are placed with their initial points at the origin.

(b) They are not linearly independent since $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ line in the same plane when they are placed with their initial points at the origin.

20. By using appropriate identities, where required, determine which of the following sets of vectors in $F(-\infty, \infty)$ are linearly dependent.

(a) $6, 3\sin^2 x, 2\cos^2 x$

(b) $x, \cos x$

(c) $1, \sin x, \sin 2x$

(d) $\cos 2x, \sin^2 x, \cos^2 x$

(e) $(3-x)^2, x^2-6x, 5$

(f) $0, \cos^3 \pi x, \sin^5 3\pi x$

21. The functions $f_1(x) = x$ and $f_2(x) = \cos x$ are linearly independent in $F(-\infty, \infty)$ because neither function is a scalar multiple of the other. Confirm the linear independence using Wroński's test.

Answer:

$W(x) = -x\sin x - \cos x \neq 0$ for some x.

22. The functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are linearly independent in $F(-\infty, \infty)$ because neither function is a scalar multiple of the other. Confirm the linear independence using Wroński's test.

23. ***(Calculus required)*** Use the Wronskian to show that the following sets of vectors are linearly independent.

(a) $1, x, e^x$

(b) $1, x, x^2$

Answer:

(a) $W(x) = e^x \neq 0$

(b) $W(x) = 2 \neq 0$

24. Show that the functions $f_1(x) = e^x$, $f_2(x) = xe^x$, and $f_3(x) = x^2e^x$ are linearly independent.

25. Show that the functions $f_1(x) = \sin x$, $f_2(x) = \cos x$, and $f_3(x) = x\cos x$ are linearly independent.

Answer:

$W(x) = 2\sin x \neq 0$ for some x.

26. Use part (a) of Theorem 4.3.1 to prove part (*b*).

27. Prove part (*b*) of Theorem 4.3.2.

28. (a) In Example 1 we showed that the mutually orthogonal vectors **i**, **j**, and **k** form a linearly independent set of vectors in R^3. Do you think that every set of three nonzero mutually orthogonal vectors in R^3 is linearly independent? Justify your conclusion with a geometric argument.

(b) Justify your conclusion with an algebraic argument. [*Hint:* Use dot products.]

True-False Exercises

In parts (a)–(h) determine whether the statement is true or false, and justify your answer.

(a) A set containing a single vector is linearly independent.

Answer:

False

(b) The set of vectors $\{\mathbf{v}, k\mathbf{v}\}$ is linearly dependent for every scalar *k*.

Answer:

True

(c) Every linearly dependent set contains the zero vector.

Answer:

False

(d) If the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then $\{k\mathbf{v}_1, k\mathbf{v}_2, k\mathbf{v}_3\}$ is also linearly independent for every nonzero scalar *k*.

Answer:

True

(e) If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent nonzero vectors, then at least one vector $\mathbf{v}_k$ is a unique linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$

Answer:

True

(f) The set of 2×2 matrices that contain exactly two 1's and two 0's is a linearly independent set in M_{22}.

Answer:

False

(g) The three polynomials $(x-1)(x+2)$, $x(x+2)$, and $x(x-1)$ are linearly independent.

Answer:

True

(h) The functions f_1 and f_2 are linearly dependent if there is a real number x so that $k_1 f_1(x) + k_2 f_2(x) = 0$ for some scalars k_1 and k_2.

Answer:

False