- Zero subspace
- Examples of subspaces
- Linear combination
- Span
- Solution space

Skills

- Determine whether a subset of a vector space is a subspace.
- Show that a subset of a vector space is a subspace.
- Show that a nonempty subset of a vector space is not a subspace by demonstrating that the set is either not closed under addition or not closed under scalar multiplication.
- Given a set *S* of vectors in R^n and a vector **v** in R^n, determine whether v is a linear combination of the vectors in *S*.
- Given a set *S* of vectors in R^n, determine whether the vectors in *S* span R^n.
- Determine whether two nonempty sets of vectors in a vector space *V* span the same subspace of *V*.

Exercise Set 4.2

1. Use Theorem 4.2.1 to determine which of the following are subspaces of R^3.

(a) All vectors of the form (*a*, 0, 0).

(b) All vectors of the form (*a*, 1, 1).

(c) All vectors of the form (*a, b, c*), where $b = a + c$.

(d) All vectors of the form (*a, b, c*), where $b = a + c + 1$.

(e) All vectors of the form (*a, b*, 0).

Answer:

(a), (c), (e)

2. Use Theorem 4.2.1 to determine which of the following are subspaces of M_{nn}.

(a) The set of all diagonal $n \times n$ matrices.

(b) The set of all $n \times n$ matrices *A* such that $\det(A) = 0$.

(c) The set of all $n \times n$ matrices *A* such that $\mathrm{tr}(A) = 0$.

(d) The set of all symmetric $n \times n$ matrices.

(e) The set of all $n \times n$ matrices *A* such that $A^T = -A$.

(f) The set of all $n \times n$ matrices *A* for which $A\mathbf{x} = 0$ has only the trivial solution.

(g) The set of all $n \times n$ matrices A such that $AB = BA$ for some fixed $n \times n$ matrix *B*.

3. Use Theorem 4.2.1 to determine which of the following are subspaces of P_3.

(a) All polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$.

(b) All polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 + a_1 + a_2 + a_3 = 0$.

(c) All polynomials of the form $a_0 + a_1x + a_2x^2 + a_3x^3$ in which a_0, a_1, a_2, and a_3 are integers.

(d) All polynomials of the form $a_0 + a_1x$, where a_0 and a_1 are real numbers.

Answer:

(a), (b), (d)

4. Which of the following are subspaces of $F(-\infty, \infty)$?

(a) All functions *f* in $F(-\infty, \infty)$ for which $f(0) = 0$.

(b) All functions *f* in $F(-\infty, \infty)$ for which $f(0) = 1$.

(c) All functions *f* in $F(-\infty, \infty)$ for which $f(-x) = f(x)$.

(d) All polynomials of degree 2.

5. Which of the following are subspaces of R^∞?

(a) All sequences **v** in R^∞ of the form $\mathbf{v} = (v, 0, v, 0, v, 0, \ldots)$.

(b) All sequences **v** in R^∞ of the form $\mathbf{v} = (v, 1, v, 1, v, 1, \ldots)$.

(c) All sequences **v** in R^∞ of the form $\mathbf{v} = (v, 2v, 4v, 8v, 16v, \ldots)$.

(d) All sequences in R^∞ whose components are 0 from some point on.

Answer:

(a), (c), (d)

6. A line *L* through the origin in R^3 can be represented by parametric equations of the form $x = at$, $y = bt$, and $z = ct$. Use these equations to show that *L* is a subspace of R^3 by showing that if $\mathbf{v}_1 = (x_1, y_1, z_1)$ and $\mathbf{v}_2 = (x_2, y_2, z_2)$ are points on *L* and *k* is any real number, then $k\mathbf{v}_1$ and $\mathbf{v}_1 + \mathbf{v}_2$ are also points on *L*.

7. Which of the following are linear combinations of $\mathbf{u} = (0, -2, 2)$ and $\mathbf{v} = (1, 3, -1)$?

(a) (2,2,2)

(b) (3,1,5)

(c) (0, 4, 5)

(d) (0, 0, 0)

Answer:

(a), (b), (d)

8. Express the following as linear combinations of $\mathbf{u} = (2, 1, 4)$, $\mathbf{v} = (1, -1, 3)$, and $\mathbf{w} = (3, 2, 5)$.

(a) $(-9, -7, -15)$

(b) (6,11,6)

(c) (0,0,0)

(d) (7,8,9)

9. Which of the following are linear combinations of

$$A = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}?$$

(a) $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix}$

(d) $\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$

Answer:

(a), (b), (c)

10. In each part express the vector as a linear combination of $\mathbf{p}_1 = 2 + x + 4x^2$, $\mathbf{p}_2 = 1 - x + 3x^2$, and $\mathbf{p}_3 = 3 + 2x + 5x^2$.

(a) $-9 - 7x - 15x^2$

(b) $6 + 11x + 6x^2$

(c) 0

(d) $7 + 8x + 9x^2$

11. In each part, determine whether the given vectors span R^3.

(a) $\mathbf{v}_1 = (2, 2, 2)$, $\mathbf{v}_2 = (0, 0, 3)$, $\mathbf{v}_3 = (0, 1, 1)$

(b) $\mathbf{v}_1 = (2, -1, 3)$, $\mathbf{v}_2 = (4, 1, 2)$, $\mathbf{v}_3 = (8, -1, 8)$

(c) $\mathbf{v}_1 = (3, 1, 4)$, $\mathbf{v}_2 = (2, -3, 5)$, $\mathbf{v}_3 = (5, -2, 9)$, $\mathbf{v}_4 = (1, 4, -1)$

(d) $\mathbf{v}_1 = (1, 2, 6)$, $\mathbf{v}_2 = (3, 4, 1)$, $\mathbf{v}_3 = (4, 3, 1)$, $\mathbf{v}_4 = (3, 3, 1)$

Answer:

(a) The vectors span

(b) The vectors do not span

(c) The vectors do not span

(d) The vectors span

12. Suppose that $\mathbf{v}_1 = (2, 1, 0, 3)$, $\mathbf{v}_2 = (3, -1, 5, 2)$, and $\mathbf{v}_3 = (-1, 0, 2, 1)$. Which of the following vectors are in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

(a) $(2, 3, -7, 3)$

(b) (0, 0, 0, 0)

(c) (1,1, 1, 1)

(d) $(-4, 6, -13, 4)$

13. Determine whether the following polynomials span P_2.

$$\mathbf{p}_1 = 1 - x + 2x^2, \quad \mathbf{p}_2 = 3 + x,$$
$$\mathbf{p}_3 = 5 - x + 4x^2, \quad \mathbf{p}_4 = -2 - 2x + 2x^2$$

Answer:

The polynomials do not span

14. Let $\mathbf{f} = \cos^2 x$ and $\mathbf{g} = \sin^2 x$. Which of the following lie in the space spanned by **f** and **g**?

(a) $\cos 2x$

(b) $3 + x^2$

(c) 1

(d) $\sin x$

(e) 0

15. Determine whether the solution space of the system $A\mathbf{x} = 0$ is a line through the origin, a plane through the origin, or the origin only. If it is a plane, find an equation for it. If it is a line, find parametric equations for it.

(a) $A = \begin{bmatrix} -1 & 1 & 1 \\ 3 & -1 & 0 \\ 2 & -4 & -5 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & -2 & 3 \\ -3 & 6 & 9 \\ -2 & 4 & -6 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & 2 & -6 \\ 1 & 4 & 4 \\ 3 & 10 & 6 \end{bmatrix}$

(e) $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 4 \\ 3 & 1 & 11 \end{bmatrix}$

(f) $A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ 3 & -9 & 3 \end{bmatrix}$

Answer:

(a) Line; $x = -\frac{1}{2}t,\ y = -\frac{3}{2}t,\ z = t$

(b) Line; $x = 2t,\ y = t,\ z = 0$

(c) Origin

(d) Origin

(e) Line; $x = -3t,\ y = -2t,\ z = t$

(f) Plane; $x - 3y + z = 0$

16. (***Calculus required***) Show that the following sets of functions are subspaces of $F(-\infty, \infty)$.

(a) All continuous functions on $(-\infty, \infty)$.

(b) All differentiable functions on $(-\infty, \infty)$.

(c) All differentiable functions on $(-\infty, \infty)$ that satisfy $\mathbf{f}' + 2\mathbf{f} = 0$.

17. (***Calculus required***) Show that the set of continuous functions $\mathbf{f} = f(x)$ on [*a, b*] such that

$$\int_a^b f\left(x\right)dx = 0$$

is a subspace of *C*[*a, b*].

18. Show that the solution vectors of a consistent nonhomoge- neous system of *m* linear equations in *n* unknowns do not form a subspace of R^n.

19. Prove Theorem 4.2.5.

20. Use Theorem 4.2.5 to show that the vectors $\mathbf{v}_1 = (1, 6, 4)$, $\mathbf{v}_2 = (2, 4, -1)$, $\mathbf{v}_3 = (-1, 2, 5)$, and the vectors $\mathbf{w}_1 = (1, -2, -5)$, $\mathbf{w}_2 = (0, 8, 9)$ span the same subspace of R^3.

True-False Exercises

In parts (a)–(k) determine whether the statement is true or false, and justify your answer.

(a) Every subspace of a vector space is itself a vector space.

Answer:

True

(b) Every vector space is a subspace of itself.

Answer:

True

(c) Every subset of a vector space *V* that contains the zero vector in *V* is a subspace of *V*.

Answer:

False

(d) The set R^2 is a subspace of R^3.

Answer:

False

(e) The solution set of a consistent linear system $A\mathbf{x} = \mathbf{b}$ of *m* equations in *n* unknowns is a subspace of R^n.

Answer:

False

(f) The span of any finite set of vectors in a vector space is closed under addition and scalar multiplication.

Answer:

True

(g) The intersection of any two subspaces of a vector space V is a subspace of V.

Answer:

True

(h) The union of any two subspaces of a vector space V is a subspace of V.

Answer:

False

(i) Two subsets of a vector space V that span the same subspace of V must be equal.

Answer:

False

(j) The set of upper triangular $n \times n$ matrices is a subspace of the vector space of all $n \times n$ matrices.

Answer:

True

(k) The polynomials $x - 1$, $\left(x - 1\right)^2$, and $\left(x - 1\right)^3$ span P_3.

Answer:

False