Computing the Robustness of Roots

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Abstract

Often relationships in nature are modelled as mappings between spaces. The roots or fixed points of these mappings play a crucial role in understanding their behavior. In practice, we observe only approximations of these mappings. The framework of *well groups* allows us to quantify the robustness and therefore the significance of the roots of a mapping. For the setting of mappings from an orientable m-manifold to $\mathbb{R}^m$, we prove a connection between well groups and topological degree theory. This connection allows for an efficient algorithm to compute the robustness of roots. We demonstrate the practicality of our algorithm on several examples of mappings approximated from point clouds in $\mathbb{R}^m$.

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1 Introduction

Mappings are ubiquitous in nature. The dynamics of a system are often modeled as mappings between spaces. Roots and fixed points of these mappings are particularly important for understanding the behavior of these systems. The study of mappings appear in a broad range of areas. A prominent example is that of dynamical systems, which studies maps from a space to itself. These are often described as vector fields [20] and appear in numerous applications including game theory, where fixed points appear as Nash equilibria [5, 17], control theory [19, 24], and visualization [21, 23, 26]. In physics, mappings appear as differential equations describing mechanics, electromagnetism, and heat diffusion, to name just a few.

In practice, we often do not have direct access to mappings. Rather we use observations to deduce a mapping. When constructing a map from observations, we get an incomplete and often noisy picture. For this reason, we look to infer robust information from the data. In this paper, we take a topological viewpoint using the language of *well groups* to quantify the robustness of the roots or fixed points of a map. We give an algorithm which allows us to compute this quantative measure from a sampling of the map.

Given a mapping $f : \mathbb{X} \to \mathbb{Y}$ between two topological spaces and $\mathbb{A} \subseteq \mathbb{Y}$ a subspace, the theoretical framework of well groups quantifies the robustness of each homology class of $f^{-1}(\mathbb{A})$ to perturbations of the mapping f. Loosely speaking, the robustness of a homology class is the distance from f to the closest mapping h such that $h^{-1}(\mathbb{A})$ fails to support the class. An important property of robustness is that it is stable. Unfortunately, there is yet no efficient algorithm to compute robustness for this most general setting.

In this paper, we concentrate on the robustness of roots for mappings of the form $f : \mathbb{M} \to \mathbb{R}^m$, where $\mathbb{M}$ is an orientable m-manifold. Roots appear as solutions to equations or as fixed points of vector fields on $\mathbb{R}^m$. There are three main contributions in this paper.

1. We relate well groups to the classical idea of topological degree theory.
2. A connection to degree theory leads to an algorithm for computing the robustness of roots.
3. We discuss an implementation of the algorithm and perform several experiments. We look at an example of the Lotka-Volterra model of population dynamics and then we study the evolution of fixed points in random vector fields under a smoothing operation.

Section 2 gives a brief introduction to the relevant background followed by an introduction to the notion of robustness in Section 3. Sections 4 and 5 contain the three main contributions of the paper. Section 6 ends the paper with a discussion of open questions and possible extensions.

2 Background

Our results require background in homology and degree theory. For a more thorough and formal introduction, we refer the reader to [15] for homology theory and [14, 15] for degree theory.

Homology. Homology is an algebraic language that defines the intuitive notion of holes in a topological space. Given a topological space $\mathbb{M}$, one first constructs a *chain group* $\mathsf{C}_p(\mathbb{M})$, for each integer p. The *chain complex* is a collection $\{\mathsf{C}_p(\mathbb{M})\}_p$ of chain groups connected by *boundary homomorphisms* $\partial_p : \mathsf{C}_p(\mathbb{M}) \to \mathsf{C}_{p-1}(\mathbb{M})$ such that $\partial_{p-1} \circ \partial_p = 0$. The pth dimensional *ordinary homology group* $\mathsf{H}_p(\mathbb{M})$ is the quotient group $\ker \partial_p / \operatorname{im} \partial_{p+1}$.

The reduced homology group is ordinary homology but applied to a modified chain complex. The chain group $\mathsf{C}_0(\mathbb{M})$ in the chain complex $\{\mathsf{C}_p(\mathbb{M})\}_p$ is replaced by the kernel of the map that sends each chain in $\mathsf{C}_0(\mathbb{M})$ to the sum of its coefficients. The *reduced homology group* $\tilde{\mathsf{H}}_p(\mathbb{M})$ is then calculated the usual way.

Often we will be interested not in the absolute homology of $\mathbb{M}$ but the homology of $\mathbb{M}$ relative to a subspace $\mathbb{A} \subseteq \mathbb{M}$. One first constructs a *relative chain complex* $\{\mathsf{C}_p(\mathbb{M}, \mathbb{A})\}_p$, where each *relative chain group* $\mathsf{C}_p(\mathbb{M}, \mathbb{A}) = \mathsf{C}_p(\mathbb{M})/\mathsf{C}_p(\mathbb{A})$. The boundary homomorphism of the ordinary chain complex induces a *relative boundary homomorphism* $\partial_p : \mathsf{C}_p(\mathbb{M}, \mathbb{A}) \to \mathsf{C}_{p-1}(\mathbb{M}, \mathbb{A})$ on the relative chain complex such that $\partial_{p-1} \circ \partial_p = 0$. The *relative homology group* $\mathsf{H}_p(\mathbb{M}, \mathbb{A})$ of the pair $(\mathbb{M}, \mathbb{A})$ is the quotient group $\ker \partial_p / \mathrm{im}\, \partial_{p+1}$.

Homology is a homotopy invariant covariant functor. A continuous mapping $f : \mathbb{M} \to \mathbb{N}$ induces a homomorphism $\mathsf{f}_p : \mathsf{H}_p(\mathbb{M}) \to \mathsf{H}_p(\mathbb{N})$ between the homology groups of the two spaces. If the mapping f takes the subspace $\mathbb{A} \subseteq \mathbb{M}$ into the subspace $\mathbb{B} \subseteq \mathbb{N}$, then f induces a homomorphism $\mathsf{f}_p : \mathsf{H}_p(\mathbb{M}, \mathbb{A}) \to \mathsf{H}_p(\mathbb{N}, \mathbb{B})$ between the two relative homology groups. If $h : \mathbb{M} \times [0, 1] \to \mathbb{N}$ is a homotopy taking $\mathbb{A}$ to $\mathbb{B}$, for each time t, then the homomorphism $h_t : \mathsf{H}_p(\mathbb{M}, \mathbb{A}) \to \mathsf{H}_p(\mathbb{N}, \mathbb{B})$ remains constant over t.

Our results make use of both integer Z and integer modulo two $\mathsf{Z}/2\mathsf{Z}$ coefficients. The choice of the coefficient ring is made clear in the notation. For example, the relative homology group $\mathsf{H}_p(\mathbb{M}, \mathbb{A}; \mathsf{Z})$ uses integer coefficients.

Orientation and the fundamental class. Let $\mathbb{M}$ be an m-dimensional manifold with boundary $\mathrm{Bd}\, \mathbb{M}$. The *local homology group* $\mathsf{H}_m(\mathbb{M}, \mathbb{M} - \{x\}; \mathsf{Z})$ for each point x in the interior of the manifold is isomorphic to Z. A *local orientation* μ_x at x is the choice of a genertor of $\mathsf{H}_m(\mathbb{M}, \mathbb{M} - \{x\}; \mathsf{Z})$. In other words, a local orientation is a choice of an isomorphism between $\mathsf{H}_m(\mathbb{M}, \mathbb{M} - \{x\}; \mathsf{Z})$ and Z. An *orientation* μ of $\mathbb{M}$ is an assignment of a local orientation μ_x to each interior point $x \in \mathbb{M}$ such that the following property is satisfied. If B is an open m-ball around x in the interior of $\mathbb{M}$ and $y \in B$, then there is a generator $\mu_B \in \mathsf{H}_m(\mathbb{M}, \mathbb{M} - B; \mathsf{Z}) \cong \mathsf{Z}$ such that μ_B restricted to $\mathsf{H}_m(\mathbb{M}, \mathbb{M} - \{y\}; \mathsf{Z})$ is μ_y. By restriction, we mean the image of μ_B under the homomorphism $\mathsf{i} : \mathsf{H}_m(\mathbb{M}, \mathbb{M} - B; \mathsf{Z}) \to \mathsf{H}_m(\mathbb{M}, \mathbb{M} - \{y\}; \mathsf{Z})$ induced by the inclusion of $\mathbb{M} - B$ into $\mathbb{M} - \{y\}$.

Assume an orientation μ on $\mathbb{M}$. A class $[\mathbb{M}] \in \mathsf{H}_m(\mathbb{M}, \mathrm{Bd}\, \mathbb{M}; \mathsf{Z})$ is called the *fundamental class* of $\mathbb{M}$ if $[\mathbb{M}]$ restricts to $\mu_x \in \mathsf{H}_m(\mathbb{M}, \mathbb{M} - \{x\}; \mathsf{Z})$, for each $x \in \mathbb{M}$. A fundamental class exists if and only if $\mathbb{M}$ is orientable [15, page 236].

Degree of mappings. Let $\mathbb{M}$ and $\mathbb{N}$ be compact oriented m-manifolds without boundary. A continuous mapping $f : \mathbb{M} \to \mathbb{N}$ induces a homomorphism $\mathsf{f}_m : \mathsf{H}_m(\mathbb{M}; \mathsf{Z}) \to \mathsf{H}_m(\mathbb{N}; \mathsf{Z})$. The *degree* of the mapping f is the unique integer $\deg(f)$ such that $\mathsf{f}_m([\mathbb{M}]) = \deg(f) \cdot [\mathbb{N}]$, where $[\mathbb{M}]$ and $[\mathbb{N}]$ are fundamental classes of the two manifolds. Geometrically the degree of a mapping is the number of times the mapping f winds $\mathbb{M}$ around $\mathbb{N}$ counted with sign. For example, the mapping $f : \mathbb{R}/\mathsf{Z} \to \mathbb{R}/\mathsf{Z}$ defined as $f(x) = 2x$ winds the circle around itself twice and therefore the degree of f is two or negative two depending on the choice of orientations.

Now assume $\mathbb{M}$ has a non-empty boundary and $f : \mathbb{M} \to \mathbb{N}$ takes the boundary of $\mathbb{M}$ into the, possibly empty, boundary of $\mathbb{N}$. The mapping f induces a homomorphism $\mathsf{f}_m : \mathsf{H}_m(\mathbb{M}, \mathrm{Bd}\, \mathbb{M}; \mathsf{Z}) \to \mathsf{H}_m(\mathbb{N}, \mathrm{Bd}\, \mathbb{N}; \mathsf{Z})$. If $[\mathbb{M}] \in \mathsf{H}_m(\mathbb{M}, \mathrm{Bd}\, \mathbb{M}; \mathsf{Z})$ and $[\mathbb{N}] \in \mathsf{H}_m(\mathbb{N}, \mathrm{Bd}\, \mathbb{N}; \mathsf{Z})$ are fundamental classes, the *degree* of f is the unique integer $\deg(f)$ such that $\mathsf{f}_m([\mathbb{M}]) = \deg(f) \cdot [\mathbb{N}]$.

There is a local but equivalent definition of the degree of a mapping $f : \mathbb{M} \to \mathbb{N}$. Choose a point $y \in \mathbb{N} - \mathrm{Bd}\, \mathbb{N}$ such that its inverse $f^{-1}(y)$ is a finite number of points. For each point p_i in the inverse, there is a homomorphism $f_i : \mathsf{H}_m(\mathbb{M}, \mathbb{M} - \{p_i\}; \mathsf{Z}) \to \mathsf{H}_m(\mathbb{N}, \mathbb{N} - \{y\}; \mathsf{Z})$ induced by f. Call $[p_i] \in \mathsf{H}_m(\mathbb{M}, \mathbb{M} - \{p_i\}; \mathsf{Z})$ the restriction of the fundamental class $[\mathbb{M}]$ and call $[y] \in \mathsf{H}_m(\mathbb{N}, \mathbb{N} - \{y\}; \mathsf{Z})$ the restriction of the fundamental class $[\mathbb{N}]$. The *degree* of f restricted to p_i is the unique integer $\deg(f, p_i)$ such that $\mathsf{f}_i([p_i]) = \deg(f, p_i) \cdot [y]$. The degree of the mapping f is the sum $\deg(f) = \sum_i \deg(f, p_i)$ over all p_i in the inverse $f^{-1}(y)$. The degree of a map is a homotopy invariant. If $h : \mathbb{M} \times [0, 1] \to \mathbb{N}$ is a homotopy such that h_t takes $\mathrm{Bd}\, \mathbb{M}$ into $\mathrm{Bd}\, \mathbb{N}$, for each t, then $\deg(h_t)$ remains constant over t.

3 Robustness

Consider the example of a mapping $f : \mathbb{R} \to \mathbb{R}$ shown in Figure 1. The mapping has four roots where the left two look more robust to perturbations than the right two. The algebraic language of well groups makes this observation precise. Well groups quantifies the stability of each root by

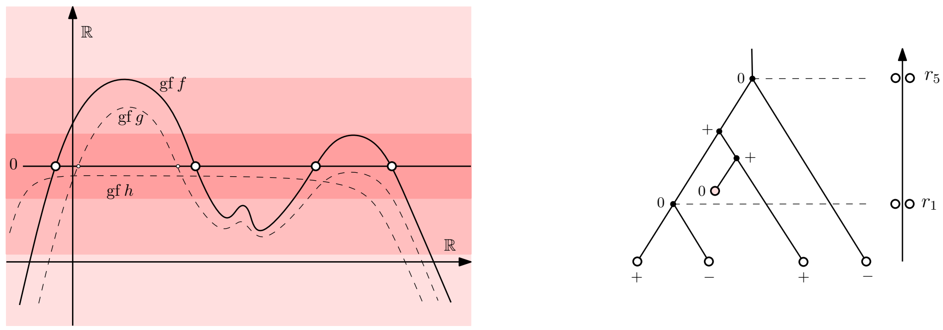


Figure 1: The mapping f has four roots. The transitions between shaded regions mark the critical values r_1 and r_5 where the rank of the well groups drop. The leaf nodes of the merge trees represent the roots of f, where the + and − signs indicate the degree of f restricted to each root. A node higher up in the tree is assigned the sum of the degrees of its leaf nodes. The well diagram, $\mathrm{Dgm}(f_0)$, of f_0 is shown to the right of the merge tree. The left two roots have robustness r_5 and the right two roots have robustness r_1.

assigning to each root a robustness. Roughly speaking, the robustness of a root is the minimum magnitude of a perturbation necessary to cancel it. An important property of robustness is that it is stable. A perturbation by an amount ε preserves roots with robustness greater than ε. We now introduce well groups in the setting of mappings from an oriented m-manifold to $\mathbb{R}^m$. For the most general definition of well groups, see [13].

Well groups. Let $\mathbb{M}$ be an orientable m-manifold and $\mathbb{R}^m$ the m-dimensional Euclidean space. A continuous mapping $h : \mathbb{M} \to \mathbb{R}^m$ is an *r-perturbation* of $f : \mathbb{M} \to \mathbb{R}^m$ if $\|f - g\|_2 = \sup_x \|f(x) - h(x)\|_2 \leq r$. Given a mapping f, define $f_0 : \mathbb{M} \to \mathbb{R}$ by setting $f_0(x) = \|f(x)\|_2$ to the norm of the image. If h is an r-perturbation of f, then $h^{-1}(0)$ includes into the *sublevel set* $\mathbb{M}_r = f_0^{-1}[0, r]$. The inclusion of spaces induces a homomorphism, $\mathsf{j}_h : \mathsf{H}_p(h^{-1}(0); \mathsf{Z}/2\mathsf{Z}) \to \mathsf{F}_p(r)$, where $\mathsf{F}_p(r)$ is shorthand for the homology group $\mathsf{H}_p(\mathbb{M}_r; \mathsf{Z}/2\mathsf{Z})$. The *well group*

$$\mathsf{U}_p(r) = \bigcap_h \operatorname{im} \mathsf{j}_h,$$

for each dimension p, is the subgroup of $\mathsf{F}_p(r)$ defined as the intersection of the images of j_h over all r-perturbations h of f. Assuming a finite number of roots, the homology groups $\mathsf{F}_p(r)$ and therefore the well groups $\mathsf{U}_p(r)$ are zero except for $p = 0$. We drop the dimension p from the notation when the dimension of interest is clear.

Death. For values $r \leq s$, the sublevel set $\mathbb{M}_r$ includes into the sublevel set $\mathbb{M}_s$. This gives us a homomorphism $\mathsf{f}_{r,s} : \mathsf{F}(r) \to \mathsf{F}(s)$ between the homology groups of the two sublevel sets. A value $r > 0$ is a *terminal critical value* of f_0 if for every sufficiently small $\delta > 0$, the homomorphism $f_{r-\delta,r+\delta}$ applied to the well group $\mathsf{U}(r - \delta)$ is not $\mathsf{U}(r + \delta)$. An important property of the well groups is that their rank can not increase with increasing radius. That is, for each choice of $r \leq s$, $\mathsf{U}(s) \subset \mathsf{f}_{r,s}(\mathsf{U}(r))$. In other words, the terminal critical values are the radii at which the well groups shrink. If $r > 0$ is not a terminal critical value, then it is a *regular value*.

Recall that we assumed the mapping f has a finite number of roots. This implies f_0 has a finite number of terminal critical values $0 = u_0 < u_1 < u_2 < \cdots < u_\ell$. For convenience, we write $\mathsf{F}(i)$ for $\mathsf{F}(u_i)$ and $\mathsf{f}_j : \mathsf{F}(0) \to \mathsf{F}(j)$ for $\mathsf{f}_{0,j}$. We say a class α in the well group $\mathsf{U}(0)$ dies a *conventional death* at u_j if $\mathsf{f}_j(\alpha) = 0$ and $\mathsf{f}_i(\alpha)$ is a non-zero class of $\mathsf{U}(i)$, for each $i < j$. We say a class $\beta \in \mathsf{U}(0)$ dies an *unconventional death* at u_j if $\mathsf{f}_j(\beta) \notin \mathsf{U}(j)$ and $\mathsf{f}_i(\beta)$ is a non-zero class of $\mathsf{U}(i)$, for each $i < j$. A class in $\mathsf{U}(0)$ lives forever or dies a single death either conventionally or unconventionally. The *robustness* of a class in $\mathsf{U}(0)$ is the value at which the class dies. See Figure 2 for an illustration of the two types of death.

Well diagrams and stability. The *well diagram* of f_0, denoted as $\mathrm{Dgm}(f_0)$, is a multiset of terminal critical values. If, at a terminal critical value, the rank of the well group drops by

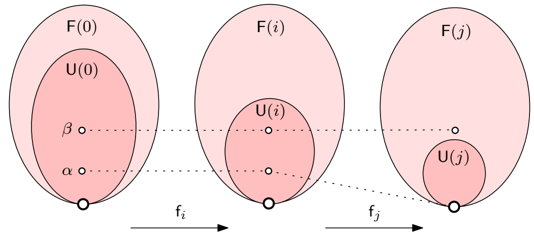


Figure 2: The class α in $\mathsf{U}(0)$ dies a conventional death whereas the class β dies an unconventional death.

k, then the value is counted k times in the well diagram. For reasons of stability, we add to the well diagram the value zero counted with infinite multiplicity.

An important property of the well diagram is that it is stable. Let $g : \mathbb{M} \to \mathbb{R}^m$ be a mapping with a finite number of roots. Order the positive points in both diagrams $\mathrm{Dgm}(f_0)$ and $\mathrm{Dgm}(g_0)$ and add zeros to make the lengths of the two sequences the same. We have

$$\begin{array}{ccccccccc} 0 & \leq & u_1 & \leq & u_2 & \leq & \ldots & \leq & u_m; \\ 0 & \leq & v_1 & \leq & v_2 & \leq & \ldots & \leq & v_m. \end{array}$$

The *bottleneck distance* between the two diagrams is defined as

$$W_\infty(\mathrm{Dgm}(f_0), \mathrm{Dgm}(f_0)) = \max_{1 \leq i \leq m} |u_i - v_i|.$$

Stability of Well Diagrams [13]. Given two mappings $f, g : \mathbb{M} \to \mathbb{R}^m$ each with a finite number of roots, the bottleneck distance between the two well diagrams is bounded by the distance between the two mappings. That is,

$$W_\infty(\mathrm{Dgm}(f_0), \mathrm{Dgm}(g_0)) \leq \sup_{x \in \mathbb{M}} \|f(x) - g(x)\|_2.$$

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous mapping such that its graph is as shown in Figure 1. The mapping $f_0 : \mathbb{R} \to \mathbb{R}$ has five critical values which we denote by the symbols $0 < r_1 < r_2 < r_3 < r_4 < r_5$. The mapping has four roots and each root contributes to the well group $\mathsf{U}(0) = \mathsf{F}(0) = (\mathsf{Z}/2\mathsf{Z})^4$. As we grow the radius from 0 to r_1, two components in the sublevel set merge and thus $\mathsf{F}(r_1) = (\mathsf{Z}/2\mathsf{Z})^3$. However, the well group $\mathsf{U}(r_1)$ drops in rank by two because there is an r_1-perturbation of f, for example the mapping g as shown in the figure, with only two roots. Growing further, the sublevel set gains a component at value r_2, looses a component at r_3, and looses another at r_4. However, an r_4 perturbation is not enough to remove the remaining two roots and therefore the rank of the well group does not change. Increasing the radius even further, the sublevel set is fully connected at radius r_5. The well group $\mathsf{U}(r_5)$ becomes trivial as there is an r_5-perturbation, for example the mapping h in the figure, with no roots. Table 1 summarizes the ranks of the homology and well groups at each radius.

Each root of f generates a class in $\mathsf{U}(0)$. Label the four classes α_1, α_2, α_3, and α_4 as their corresponding roots occur in Figure 1 from left to right. There are conventional deaths and unconventional deaths at both r_1 and r_5. At radius r_1, the sum $\alpha_3 + \alpha_4$ dies a conventional death, and the individual classes α_3 and α_4 die unconventional deaths. Similarly at radius r_5, the sum $\alpha_1 + \alpha_2$ dies a conventional death, and the individual classes α_1 and α_2 die unconventional deaths. Therefore, the robustness of the left two roots is r_5 whereas the robustness of the right two roots is r_1. Stability of the well diagram implies that any perturbation of the mapping f by an amount r_1 has at least two roots with robustness between $r_5 - r_1$ and $r_5 + r_1$. For those familiar with persistence [11], we note that this gauarantee is not implied by the stability theorem for persistence diagrams.

	$[0, r_1)$	$[r_1, r_2)$	$[r_2, r_3)$	$[r_3, r_4)$	$[r_4, r_5)$	$[r_5, \infty)$
$\mathsf{F}(r)$	4	3	4	3	2	1
$\mathsf{U}(r)$	4	2	2	2	2	0

Table 1: Shown are ranks of homology and well groups for the example given in Figure 1.

4 Algorithm

Well groups quantify stability by assigning to each root its robustness. Thus far, there is no obvious algorithm to compute robustness, especially since the definition of the well group involves the intersection of an infinite number of images. In this section, we use degree theory to derive an equivalent definition of the well group leading to a practical algorithm. The main theorem in this section is the Equivalence Theorem. The Equivalence Theorem relates the degree of a mapping to the rank of a well group.

An equivalent definition. Let $f : \mathbb{M} \to \mathbb{R}^m$ be a mapping from a compact orientable m-manifold to $\mathbb{R}^m$ and assume the number of roots of f is finite. Recall that the mapping $f_0 : \mathbb{M} \to \mathbb{R}$ assigns to each point x in the manifold the Euclidean norm of $f(x)$. For $r \geq 0$, the sublevel set $\mathbb{M}_r = f_0^{-1}[0, r]$.

Choose a regular value $r > 0$ and let C be a connected component of $\mathbb{M}_r$. Assuming $\mathbb{M}_r$ is an m-manifold, the fundamental class $[\mathbb{M}]$ of $\mathbb{M}$ restricts to a fundamental class $[C]$ of C. Using the long exact sequence of a pair [15, page 115],

$$\longrightarrow \mathsf{H}_m(C; \mathsf{Z}) \longrightarrow \mathsf{H}_m(C, \operatorname{Bd} C; \mathsf{Z}) \xrightarrow{\mathsf{b}} \tilde{\mathsf{H}}_{m-1}(\operatorname{Bd} C; \mathsf{Z}) \longrightarrow$$

$[\mathbb{M}]$ restricts even further to a fundamental class of $\operatorname{Bd} C$. The homomorphism $\mathsf{b} : \mathsf{H}_m(C, \operatorname{Bd} C) \to \tilde{\mathsf{H}}_{m-1}(\operatorname{Bd} C)$ takes a class $[\alpha]$ to $[\partial_m \alpha]$. The class $[\operatorname{Bd} C] = \mathsf{b}([C])$ is the restriction of the fundamental class of C to a fundamental class of $\operatorname{Bd} C$. Letting $[\Delta] \in \mathsf{H}_m(\mathbb{B}_r, \operatorname{Bd} \mathbb{B}_r)$ be a fundamental class consistent with the orientation on $\mathbb{R}^m$, the boundary map in the long exact sequence of a pair restricts $[\Delta]$ to a fundamental class $[\Delta]_{\operatorname{Bd}} \in \tilde{\mathsf{H}}_{m-1}(\mathbb{S}^{m-1})$ of the $(m-1)$-sphere bounding the ball.

Assume for now, $C \cap \operatorname{Bd} \mathbb{M} = \emptyset$. The restriction of f to C and $\operatorname{Bd} C$ induces the homomorphisms $\mathsf{f_C} : \mathsf{H}_m(C, \operatorname{Bd} C; \mathsf{Z}) \to \mathsf{H}_m(\mathbb{B}_r, \operatorname{Bd} \mathbb{B}_r; \mathsf{Z})$ and $\mathsf{f_{Bd\,C}} : \tilde{\mathsf{H}}_{m-1}(\operatorname{Bd} C; \mathsf{Z}) \to \tilde{\mathsf{H}}_{m-1}(\mathbb{S}^{m-1}; \mathsf{Z})$. The degree of f restricted to C is the unique integer $\deg(f, C)$ such that $\mathsf{f_C}([C]) = \deg(f, C) \cdot [\Delta]$. The degree of f restricted to $\operatorname{Bd} C$ is the unique integer $\deg(f, \operatorname{Bd} C)$ such that $\mathsf{f_{Bd\,C}}([\operatorname{Bd} C]) = \deg(f, \operatorname{Bd} C) \cdot [\Delta]_{\operatorname{Bd}}$.

Degree Lemma. $\deg(f, C) = \deg(f, \operatorname{Bd} C)$.

Proof. Consider the following commutative diagram.

$$\begin{array}{ccc} \mathsf{H}_m(C, \operatorname{Bd} C; \mathsf{Z}) & \xrightarrow{\mathsf{b}} & \tilde{\mathsf{H}}_{m-1}(\operatorname{Bd} C; \mathsf{Z}) \\ \downarrow \mathsf{f_C} & & \downarrow \mathsf{f_{Bd\,C}} \\ \mathsf{H}_m(\mathbb{B}_r, \operatorname{Bd} \mathbb{B}_r; \mathsf{Z}) & \xrightarrow[\mathsf{c}]{\cong} & \tilde{\mathsf{H}}_{m-1}(\mathbb{S}^{m-1}; \mathsf{Z}) \end{array}$$

The homomorphisms b and c are the boundary homomorphisms from the long exact sequence of a pair. We have $\mathsf{c}(\mathsf{f_C}([C])) = \mathsf{c}(\deg(\mathsf{f}, C) \cdot [\Delta]) = \deg(\mathsf{f}, C) \cdot \mathsf{c}([\Delta]) = \deg(\mathsf{f}, C) \cdot [\Delta]_{\operatorname{Bd}}$ and $\mathsf{f_{Bd\,C}}(\mathsf{b}([C])) = \mathsf{f_{Bd\,C}}([\operatorname{Bd} C]) = \deg(f, \operatorname{Bd} C) \cdot [\Delta]_{\operatorname{Bd}}$. Therefore $\deg(f, C) = \deg(f, \operatorname{Bd} C)$. ◻

For the boundary case $C \cap \operatorname{Bd} \mathbb{M} \neq \emptyset$, the homomorphism $\mathsf{f_C}$ is not well-defined. We handle this case by setting $\deg(f, C) = 0$.

Equivalence Theorem. If r is a regular value and $\mathbb{M}_r$ an m-manifold, then the rank of the well group $\mathsf{U}(r)$ is the number of components C of $\mathbb{M}_r$ such that $\deg(f, C) \neq 0$.

Proof. The choice of a regular value r implies that the well groups $\mathsf{U}(r-\delta)$ through $\mathsf{U}(r+\delta)$ are isomorphic, for all sufficiently small $\delta > 0$.

First, we consider the case $\deg(f, C) \neq 0$. We have the following commutative diagram for any ρ-perturbation h of f, for $\rho < r$.

$$\begin{array}{ccc} \mathsf{H}_m(C, \mathrm{Bd}\, C; \mathsf{Z}) & \xrightarrow{\mathsf{j}} & \mathsf{H}_m(C, C - h^{-1}(0); \mathsf{Z}) \\ \downarrow \mathsf{f_C} & & \downarrow \mathsf{h_C} \\ \mathsf{H}_m(\mathbb{B}_r, \mathrm{Bd}\, \mathbb{B}_r; \mathsf{Z}) & \xrightarrow[\mathsf{i}]{\cong} & \mathsf{H}_m(\mathbb{R}^m, \mathbb{R}^m - \{0\}; \mathsf{Z}) \end{array}$$

The horizontal maps are induced by inclusion of spaces and the vertical maps by f and h. By the hypothesis, the class $\mathsf{f_C}([C]) \neq 0$ and therefore $\mathsf{h_C}(\mathsf{j}([C])) \neq 0$. This means the number of points in the intersection $h^{-1}(0) \cap C$ is at least the absolute value of $\deg(f, C)$. Therefore the class in $\mathsf{H}_0(\mathbb{M}_r)$ generated by C belongs to the well group $\mathsf{U}(r)$.

If $\deg(f, C) = 0$, we prove the existence of an $(r+\rho)$-perturbation h of f such that $h^{-1}(0) \cap C = \varnothing$, for all sufficiently small $\rho > 0$. The existence of such a perturbation implies that C does not contribute to the well group $\mathsf{U}(r)$.

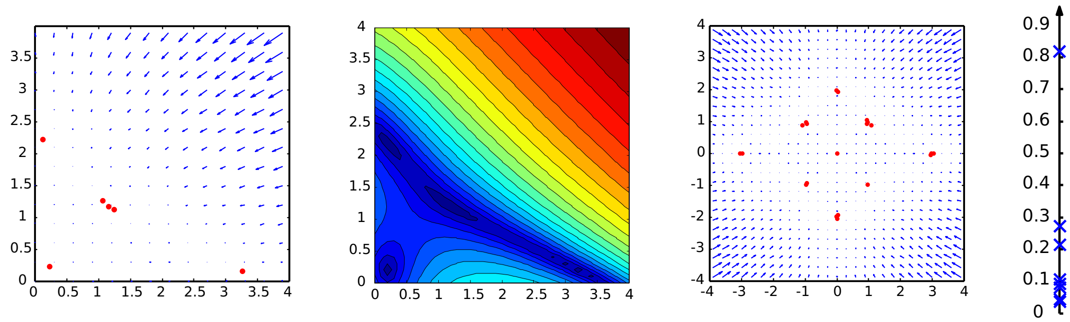
Assume first, $C \cap \mathrm{Bd}\, \mathbb{M} = \varnothing$. By the Degree Lemma, $\deg(f, C) = \deg(f, \mathrm{Bd}\, C) = 0$. The Hopf Extension Theorem [14, 16] says that if the mapping $f : \mathrm{Bd}\, C \to \mathbb{S}^{m-1}$ has degree zero, then there is an extension $g : C \to \mathbb{S}^{m-1}$ to C such that g restricted to $\mathrm{Bd}\, C$ is f. Define the $(r+\rho)$-perturbation h of f as $h = (0.5 - \rho) \cdot f + (0.5 + \rho) \cdot g$. The preimage $h^{-1}(0)$ does not intersect C as required.

We now handle the boundary case $C \cap \mathrm{Bd}\, \mathbb{M} \neq \varnothing$. The Degree Lemma is unavailable in this case but we resolve this problem with a classical trick. A *double* of $\mathbb{M}$ is two disjoint copies of $\mathbb{M}$ with opposite orientations glued together along their boundaries using the identity map. In other words, the double of $\mathbb{M}$ is $\mathbb{W} = \mathbb{M} \sqcup_{\mathrm{id}} -\mathbb{M}$. The mapping f on $\mathbb{M}$ extends to a mapping $\hat{f} : \mathbb{W} \to \mathbb{R}^m$. Let $U \subseteq \mathbb{W}$ be the subspace of the double covering C twice. The mapping $\hat{f}$ takes U to $\mathbb{B}_r$ and the boundary of U to the boundary of $\mathbb{B}_r$. The degree $\deg(\hat{f}, U)$ of $\hat{f}$ restricted to U is zero because for every point in $p \in \hat{h}^{-1}(0)$ there is a point in $q \in \hat{h}^{-1}(0)$ such that the degree of $\hat{h}$ restricted to p has a sign opposite to the degree of $\hat{h}$ restricted to q. An employment of the Degree Lemma and the Hopf Extension Theorem results in an $(r+\rho)$-perturbation $\hat{h}$ of $\hat{f}$ such that $\hat{h}^{-1}(0) \cap U = \varnothing$. The perturbation $\hat{h}$ restricts to an $(r+\rho)$-perturbation $h : \mathbb{M} \to \mathbb{R}^m$ of f. The preimage $h^{-1}(0)$ intersect C is empty as required. ◻

The Equivalence Theorem says that by keeping track of the degrees of each component in the growing sublevel set, we are also keeping track of the rank of the well group.

Merge tree. At radius $r = 0$, the sublevel set $\mathbb{M}_0$ is the set of roots of f. As r grows, components within the sublevel set merge and new ones may be born. We track this history using a tree. Each node of the tree presents the birth of a component or the merger of two or more components. At radius zero, we add, to the tree, a node for each root of f. As r increases, we trace each component along an edge of the tree. See Figure 1 for an example.

To each node of the tree, we record the degree of the mapping f restricted to the connected component that node represents. We start at the bottom most nodes of the tree representing the roots of f. Assign to the node presenting a root p_i the degree $\deg(f, p_i)$ of f restricted to p_i. Note $\deg(f, p_i)$ is plus one or negative one. Now consider a node higher up on the tree. The node represents a connected component $C \subseteq \mathbb{M}_r$ in a sublevel set $\mathbb{M}_r$, for some $r > 0$. The component C may not be an m-manifold, but we can go higher up the tree by an arbitrarily



(a) (b) (c) (d)

Figure 3: (a) A plot of the Lotka-Volterra competative equations with a small shift in the coordinates represented by vectors, with the detected fixed points shown. (b) The contour plot of the norm of the function. (c) The Lotka-Volterra equations with the upper right quandrant tiled and reflected appropriately. (d) The well diagram for (c).

small amount so that C grows into an m-manifold C^δ. This is certainly the case for smooth and piecewise linear mappings. If $C^\delta \cap \mathrm{Bd}\,\mathbb{M} = \varnothing$, the mapping f takes C^δ to $\mathbb{B}_{r+\delta}$ and the boundary of C^δ to the boundary of the ball. The degree $\deg(f, C^\delta)$ of the mapping f restricted to C^δ is the sum of the local degrees $\deg(f, p_i)$ over each root $p_i \in C^\delta$. Assign this sum to the node in the tree representing C. If $C^\delta \cap \mathrm{Bd}\,\mathbb{M} \neq \varnothing$, simply assign to the node a degree of zero as the component C does not contribute to the well group.

An initial computation of the degree of f at each root is sufficient to determine the degree of any component of a sublevel set. By the Equivalence Theorem, a component of a sublevel set contributes to a rank to the well group iff its degree is non-zero. The robustness of a root is the height of its lowest degree zero ancestor.

Towards an implementation. We conclude this section by stating a lemma which allows us to move from the continuous setting to a point cloud. Let $\mathbb{M} \subseteq \mathbb{R}^m$ be an m-dimensional compact submanifold of $\mathbb{R}^m$ and $f : \mathbb{M} \to \mathbb{R}^m$ a c-Lipschitz mapping. Given a triangulation K of $\mathbb{M}$ and f values at each of its vertices, we linearly interpolate over its simplices resulting in a continuous mapping $\hat{f} : |K| \to \mathbb{R}^m$. If the vertices of K are an ε-sampling, then $\|f - \hat{f}\|_2 \leq c\varepsilon$. The following is a consequence of the Stability of Well Diagrams.

Triangulation Lemma. The bottleneck distance between the well diagrams of f and the piecewise interpolation $\hat{f}$ is bounded by

$$W_\infty(\mathrm{Dgm}(f), \mathrm{Dgm}(\hat{f})) \leq c\epsilon.$$

5 Experiments

Implementation In this section, we describe an implementation of the algorithm described in Section 4. As input, we take a point cloud embedded in $\mathbb{R}^m$ along with a mapping that assigns to each point a vector in $\mathbb{R}^m$. The algorithm has several initialization steps:

1. Build a triangulation over the point cloud and then linearly interpolate so that we have a continuous mapping from the underlying space of the triangulation to $\mathbb{R}^m$.
2. Identity the roots of the PL mapping.
3. Compute the degree of the PL map restricted to each root.

As our examples are in low dimensions, we use the CGAL library [1] to build a Delaunay triangulation. The complexity of the Delaunay triangulation is $O(n^{\lceil \frac{m}{2} \rceil})$, where n is the number

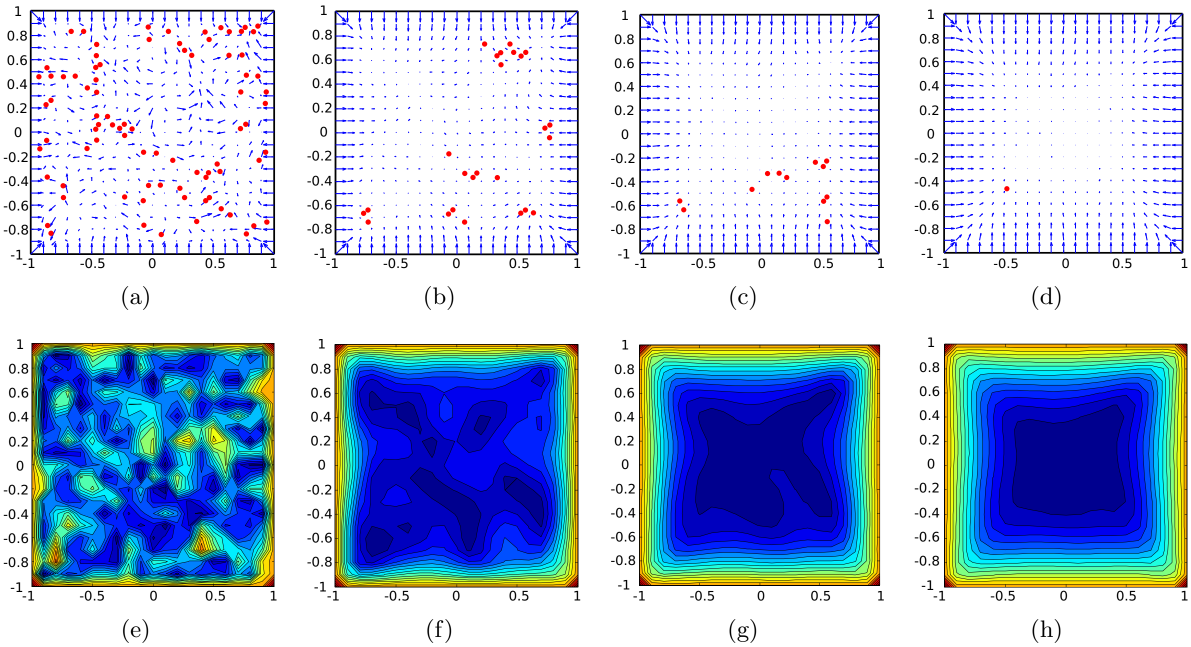


Figure 4: (a) A random vector field with the boundary vectors pointing inwards. The vector field after smoothing (b) 5 times, (c) 10 times, and (d) 15 times. (e) The norm of the random vector field with the boundary vectors pointing inwards. The norm after smoothing (f) 5 times, (g) 10 times, and (h) 15 times

of points in the point cloud. This, as well as numerical problems, limit this approach to low dimensional data. However, if the data lies on a grid, one may construct a triangulation combinatorially, allowing for analysis in higher dimensions. Given the triangulation, steps 2 and 3 can be performed in time that is linear in the number of top dimensional simplices. To find the roots of the PL mapping, we check whether the origin is contained in the image of each top dimensional simplex. This check requires solving a system of m linear equations each of m variables. For this, we use the Armadillo C++ library [18][1]. If s is the number of maximal simplices the cost of identifying all the roots is $O(sm^{\omega})$, where ω is the cost of matrix multiplication [7]. The third step involves computing orientations which require determinants. If an m-simplex contains a root, we check whether the orientation of the m-simplex is preserved when mapped to $\mathbb{R}^m$ by comparing the signs of two determinants. If the two determinants have the same sign, the sign of the root is $+1$, otherwise it is -1. The cost of each determinant is the same as multiplying two matrices [7].

The total cost of steps 2 and 3 is $O(sm^{\omega})$. The well diagram is computed using the merge tree. This is done with a modified union-find algorithm reminiscent of the 0-dimensional persistent homology computation [10]. Generically, the roots of the PL mapping are isolated and lie in the interior of m-simplices. We begin by modifying the triangulation by adding the roots of the mapping as vertices to the triangulation. Perform a `MakeSet` for each these vertices and assign to each set a value containing its sign computed in step 3. Now sort the vertices of the triangulation according to the norm of the PL mapping $\hat{f}$. We process each vertex in order of increasing norm values. For each vertex v, perform a `Find` on v and then a `Union` on each set containing a vertex in the lower link of v. Assign to the new set the sum of the degrees of each set in the lower link of v. If the lower link of v contains two disjoint sets both with non-zero degree, then there is a conventional death and we mark this event with a point in the well diagram at a value that is the norm of $\hat{f}(v)$. If the merger of two sets both with non-zero degree results in a set with degree zero, we also have an unconventional death which we mark

[1]Armadillo uses the standard LAPACK [2] and ATLAS [22] libraries to perform the linear algebra operations.

with a point in the well diagram. If more than two sets merge at v, we treat the situation as multiple pairs of mergers. The running time is $O(e\alpha(e))$, where e is the number of edges in the modified triangulation[2] and $\alpha(\cdot)$ is the inverse Ackermann function. The algorithm requires $O(e)$ storage. Clearly, steps 1, 2, and 3 are the pacing phases of the computation.

With regards to the boundary, we assume that the region over which the mapping is defined is convex. In this case, it is simple to mark the boundary vertices as such and set the degree of a set to zero when it encounters the boundary. In the more general case of embedded manifolds, the boundaries must either be given to us or inferred from the point cloud which is a research question in its own right.

Population Dynamics We now consider an example from population dynamics, in paritic-ular the classic example of Lotka-Volterra model of competition (Section 6.4 [20]). This model decribes two species x and y competing for a common limited resource. The assumptions are: each would grow to its carrying limit (determined by the limited resource) in the absence of the other species and we assume that the growth rate of each species is decreased by the presence of the other. These are described by the following equations:

$$\dot{x} = x(3 - x - 2y), \qquad \dot{y} = y(2 - x - y). \tag{1}$$

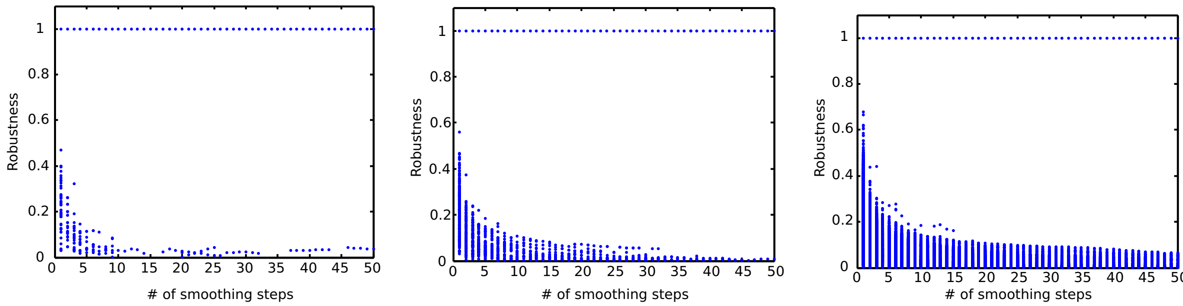
This system has four fixed points, which are: a source at $(0, 0)$, two sinks at $(0, 2), (3, 0)$, and a saddle point at$(1, 1)$. In the sampled case (Figures 3(a) and (b)), we see that several non-robust fixed points appear due to numerical errors. An important point in this example is that three of the fixed points are on the boundary, since the population of a species cannot be negative. For numerical reasons, we perturb the above equations to move the fixed points off of the boundary (which was done in the Figures 3(a) and (b)). In this case the only robust fixed point is $(1, 1)$ as the others are almost immediately set to degree 0 as we reach the boundary. However, we give a heuristic method of studying this case. By tiling and appropriately reflecting the vector field the fixed points are no longer near the boundary (Figure 3(c)). Note that in this case the mapping is still continuous and the perturbation shown in Figures 3(a) and (b) are not required. Although this tiling introduces multiple copies of the fixed points, it does allow us to measure the robustness of the fixed points on the boundary(Figure 3(d)).

To interpret the tiled case, we note that the fixed points die in a symmetric manner in all four quadrants. By looking at the location of the fixed points associated with the points in the well diagram, we see that the fixed points $(1, 1)$ and $(0, 2)$ die at 0.2. This is followed by the death of the fixed point $(3, 0)$ at 0.27. What remains is the fixed point at the origin which dies when we hit the boundary at around 0.8. If we consider this fixed point in the original, non-tiled case, it can be seen that the only way to remove this fixed point is change the vector field so that it flows away from the origin everywhere. Therefore, it is natural and informative that the fixed point at the origin dies only when we hit the boundary.

Despite the numerical instability which introduces additional fixed points, we see that by computing the robustness, we can recover the four real fixed points in the underlying system. In this case, the multiplicities in the well diagram are especially important to take into account. Intuitively, we can filter out the fixed points which are introduced by numerical error. Furthermore, we see that the three robust fixed points away from the origin have unequal robustness, giving an insight into the dynamics involved. We believe that this method of tiling to deal with fixed points on the boundary may work in a more general setting. However, a theoretical investigation into approach is beyond the scope of this paper.

Random Vector Fields Our second example is motivated by heat diffusion. More specifically, we are interested in seeing how the well diagram of a vector field behaves under a smoothing operation. We construct a vector field on the unit square. First, assign to each point on the

[2]This is the number of edges in the original triangulation plus $(m + 1)$ times the number of roots.



(a) (b) (c)

Figure 5: The well diagrams for a (a) 20×20 grid, (b) 40×40 grid, and a (c) 200×200 grid.

boundary a unit length vector pointing to the center of the square. To each interior grid point, we assign a normally distributed vector. Keeping the vector on the boundary fixed, we smooth by assigning to each grid point an average of its neighboring vectors. Of course, we triangulate and linearly interpolate so we may find the zeros of this vector field and compute its well diagram. Given the boundary condition, we expect to see at least one fixed point in the interior of the square. See Figure 4.

Shown in Figure 5 are the well diagrams generated after each of the fifty smoothing steps. The rate at which the points in the well diagrams go to zero depends on the density of the grid points in the unit square. Shown are a sequence of well diagrams for three grid point densities. In all cases, we end up with one robust fixed point as expected. Interestingly, non-robust points do not completely disappear. As can be seen in Figure 5(a), points with low robustness repeatedly appear and disappear after significant smoothing. We believe this phenomenon deserves further investigation.

6 Discussion

The framework of well groups is very general but consistent with the spirit of persistent homology. As of now, there are no efficient algorithms to compute the well groups for the general case. For mappings to the reals, the rank of well groups are encoded in the extended persistence diagram [4]. In this paper, we have shown how to apply well groups to quantify the robustness of roots of mappings and compute them in the case of mappings from an orientable m-manifold to $\mathbb{R}^m$. We have shown experimental results for $m = 2$ illustrating the practicality of our algorithm.

A number of open questions remain. Our algorithm depends on a triangulation. An interesting question is whether we can we recover the degree information from the point cloud directly using topological reconstruction techniques [6, 8, 9]. This would allow us to skip the initialization steps in the algorithm and compute robustness using easier to compute complexes, such as the Vietoris-Rips.

When visualizing vector fields, it is often important to simplify the data first [25]. The robustness of a fixed point allows one to judge its importance. Given a parameter ε, is there an efficient algorithm to construct a vector field no more than ε away from the original such that its well diagram has no points below ε? Such an approach to the simplification of real valued functions using persistence diagrams already exists [3, 12].

Our experiments provide a proof-of-concept showing that robustness is not only computable but contains useful information. Our method distinguishes between roots which arise due to noise or numerical instability from those which are intrinsic to the system. In the future, we will apply this framework to other settings including game theory, numerical analysis, and other dynamical systems of interest.

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