1 (12 points) Define the vectors $\vec{u}, \vec{v}$, and $\vec{w}$, and the matrices A, B, and C, as follows:

$$\vec{u} = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 3 & -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 2 \\ 3 & 1 \\ 2 & 0 \\ 1 & 4 \end{bmatrix}.$$

If the following quantities are defined, compute them; if not, explain why (points are 1/1/1/1/2/2/4):

(a) $A\vec{u}$.

Solution: $\begin{bmatrix} -1+6+1+0 \\ -2+9+2+0 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$.

(b) $B\vec{v}$.

Solution: $\begin{bmatrix} 0-6+7 \\ 1-9+28 \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \end{bmatrix}$.

(c) AC.

Solution: $\begin{bmatrix} -1+6-2+0 & 2+2+0+0 \\ -2+9-4+1 & 4+3+0+4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & 11 \end{bmatrix}$.

(d) BC.

Solution: It's not defined; B is a 2×3 matrix and C is a 4×2 matrix and $3 \neq 4$.

2 (12 points) Complete the following definitions:

(a) A set of vectors $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ in $\mathbb{R}^n$ is *linearly independent* provided that none of the vectors $\vec{v_i}$ is a linear combination of the others.

(b) The *span* of a set of vectors $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ in $\mathbb{R}^n$ is the set of all linear combinations of those vectors.

(c) The *null space* of an $m \times n$ matrix A is the set of vectors $\vec{x} \in \mathbb{R}^n$ for which $A\vec{x} = \vec{0}$.

(d) The *rank* of an $m \times n$ matrix A is the dimension of the column space $C(A)$.

(e) A *basis* for a subspace V of $\mathbb{R}^n$ is a set of vectors $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ in V with the following properties: the set of vectors is a linearly independent set and the vectors in the set span V.

(f) A *linear transformation* is a function T from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that the following two properties hold: for all $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$ and any scalar c, $T(\vec{x}+\vec{y}) = T(\vec{x})+T(\vec{y})$ and $T(c\vec{x}) = cT(\vec{x})$.

3 (12 points)

(a) Is each of these two subsets a subspace of $\mathbb{R}^2$? Explain why or why not.

i. The first quadrant; that is, the set of points (x, y) with $x \geq 0$ and $y \geq 0$.
Solution: This is not a subspace, because it is not closed under scalar multiplication. For example, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in the first quadrant, but its scalar multiple $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ is not.

ii. The line $y = 2x + 1$.
Solution: This is not a subspace because it does not contain the zero vector!

(b) Is each of these two functions a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$? Explain why or why not, and if it is a linear transformation, give the matrix:

i. $T(x_1, x_2) = (e^{x_1}, x_1 + x_2)$.
Solution: This is not a linear transformation; for example, $T(1, 0) = (e, 0)$, so $T(1, 0) + T(1, 0) = (2e, 0)$. However, $T((1, 0) + (1, 0)) = T(2, 0) = (e^2, 0)$, which is different from $T(1, 0) + T(1, 0) = T(2, 0)$.

ii. $T(x_1, x_2) = (2x_1 + 3x_2, 7x_1 + 8x_2)$.
Solution: This is a linear transformation: let (x_1, x_2) and (y_1, y_2) be vectors (written as points) and let c be a scalar. Then:

$$T((x_1, x_2) + (y_1, y_2)) = (2(x_1 + y_1) + 3(x_2 + y_2), 7(x_1 + y_1) + 8(x_2 + y_2))$$

$$= (2x_1 + 3x_2, 7x_1 + 8x_2) + (2y_1 + 3y_2, 7y_1 + 8y_2) = T(x_1, x_2) + T(y_1, y_2);$$

$$T(c(x_1, x_2)) = (2cx_1 + 3cx_2, 7cx_1 + 8cx_2) = c(2x_1 + 3x_2, 7x_1 + 8x_2) = cT(x_1, x_2).$$

The matrix of the linear transformation is $\begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix}$.

4 (10 points)

(a) Find a parametric representation for the line in $\mathbb{R}^4$ which passes through the points $(1, 2, -1, 2)$ and $(3, -2, 1, 0)$.

Solution: A vector parallel to the line is

$$\begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 2 \\ -2 \end{bmatrix}.$$

So a parametric representation for the line is

$$\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ -4 \\ 2 \\ -2 \end{bmatrix} : t \in \mathbb{R}\}.$$

(b) Find a parametric representation for the plane in $\mathbb{R}^3$ which passes through the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

solution: Two vectors parallel to the plane are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. These are linearly independent. So a parametric representation for the plane is:

$$\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} : t, s \in \mathbb{R}.\}$$

5 (10 points) Compute the reduced row echelon form of the following matrix:

$$A = \begin{bmatrix} 0 & 1 & -1 & 0 & 2 \\ 3 & 3 & 6 & 0 & 6 \\ 1 & 4 & -1 & -3 & -1 \\ -1 & 1 & -4 & 1 & 5 \end{bmatrix}.$$

Solution: First switch the first and second rows, then divide the new first row by 3:

$$\begin{bmatrix} 1 & 1 & 2 & 0 & 2 \\ 0 & 1 & -1 & 0 & 2 \\ 1 & 4 & -1 & -3 & -1 \\ -1 & 1 & -4 & 1 & 5 \end{bmatrix}.$$

Subtract the first row from the third row, and add the first row to the fourth row, to get:

$$\begin{bmatrix} 1 & 1 & 2 & 0 & 2 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 3 & -3 & -3 & -3 \\ 0 & 2 & -2 & 1 & 7 \end{bmatrix}.$$

Subtract the second row from the first row, subtract three times the second row from the third row, and subtract twice the second row from the fourth row, to get:

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & -3 & -9 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}.$$

Divide the fourth row by -3 and then subtract it from the fifth row to get the final answer of:

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

6 (15 points) Consider the following matrix A and its reduced row echelon form rref(A); you do not need to check that the row reduction is correct:

$$A = \begin{bmatrix} 1 & 6 & -4 & 2 \\ 2 & -1 & 5 & 3 \\ -1 & -4 & 2 & 4 \\ 3 & 5 & 1 & 1 \\ -2 & -3 & -1 & 7 \end{bmatrix}; \; rref(A) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find a basis for the column space $C(A)$. What is the rank?

Solution: The pivot variables are in columns 1, 2, and 4, so those columns in the un-reduced matrix form a basis for the column space. There are three of them, so the rank is 3. The basis is:

$$\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ -4 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \\ 7 \end{bmatrix} \}.$$

(b) Find a basis for the null space $N(A)$. What is the nullity?

Solution: To find a basis, we solve $A\vec{x} = \vec{0}$, which is the same as solving rref $A\vec{x} = \vec{0}$. That is the system of linear equations:

$$x_1 + 2x_3 = 0, \; x_2 - x_3 = 0, \; x_4 = 0.$$

x_3 is the lone free variable, so solve for all other variables in terms of x_3:

$$x_1 = -2x_3, \; x_2 = x_3, \; x_3 = x_3, \; x_4 = 0.$$

This means:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Hence a basis for the null space $N(A)$ is $\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \}$. The nullity is 1.

(c) Let the vectors $\vec{u} \in \mathbb{R}^4$ and $\vec{b} \in \mathbb{R}^5$ be given by

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} -3 \\ 7 \\ 1 \\ 4 \\ -3 \end{bmatrix}.$$

Given that $A\vec{u} = \vec{b}$ (you do not need to check this), find the set of all solutions $\vec{x} \in \mathbb{R}^4$ to $A\vec{x} = \vec{b}$.

Solution: By Proposition 8.2, this set of all solutions is the set of all vectors of the form $\vec{x_p} + \vec{x_h}$, where $\vec{x_p}$ is a particular solution and $\vec{x_h}$ is anything in $N(A)$. In this case, u is a particular solution to $A\vec{x} = \vec{b}$, and we know the nullspace from part b). So the set of all solutions is of the form:

$$\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} : x_3 \in \mathbb{R} \}.$$

It is a line in $\mathbb{R}^4$ that does not pass through the origin.

7 (14 points)

(a) (4 points) Find the matrix of the linear transformation T, where T is the linear transformation that leaves points on the y-axis fixed, but shifts a point (x, y) off the y-axis vertically by $-x$ units. (As we mentioned in class, this sort of transformation is called a *shear*.)

Solution: This transformation sends $\vec{e}_1$, which has $x = 1$, downwards by 1 unit, so $T(\vec{e}_1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. However, $T(\vec{e}_2) = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So the matrix for T, call it A, is:

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

(b) (4 points) Find the matrices of the linear transformations R and S, where R is a rotation by $\pi/2$ radians in the *clockwise* direction and S is a rotation by $\pi/2$ radians in the counterclockwise direction. (Here S and R are said to be *inverses*.)

Solution: By our formula for rotation matrices, we can easily see that the matrix B for R and the matrix C for S are given by:

$$B = \begin{bmatrix} \cos(-\pi/2) & -\sin(-\pi/2) \\ \sin(-\pi/2) & \cos(-\pi/2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(c) (6 points) Compute the matrix of the transformation $R \circ T \circ S$. What kind of linear transformation is this?

Solution: The matrix for the transformation $R \circ T \circ S$ is given by the matrix BAC. We compute that $BA = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, so

$$BAC = (BA)C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

This corresponds to the linear transformation that preserves the x-axis but shifts points off the x-axis horizontally by y units. It is another shear transformation.

8 (15 points)

(a) Suppose that A is a $m \times n$ matrix with rank n. Show that the equation $A\vec{x} = \vec{b}$ never has more than one solution, no matter which $\vec{b} \in \mathbb{R}^m$ we choose.

Proof: Pick any $\vec{b} \in \mathbb{R}^m$. A has rank n, so by the rank-nullity theorem the nullity of A must be 0. That is, the nullspace consists only of the zero vector. If $A\vec{x} = \vec{b}$ has two solutions $\vec{x}_1$ and $\vec{x}_2$, then $A(\vec{x}_1 - \vec{x}_2) = A\vec{x}_1 - A\vec{x}_2 = \vec{b} - \vec{b} = \vec{0}$, so $\vec{x}_1 - \vec{x}_2$ is in $N(A)$. But because the nullspace is only the zero vector, that means $\vec{x}_1 - \vec{x}_2 = \vec{0}$, so $\vec{x}_1 = \vec{x}_2$. Therefore $A\vec{x} = \vec{b}$ only has one solution.

(b) Suppose that A is an $m \times n$ matrix with nullity equal to $n - 1$. Prove that any two nonzero vectors in $C(A)$ are collinear (that is, one is a scalar multiple of the other).

Proof: Again using the rank-nullity theorem, the rank of A must be equal to 1. This means that the column space is a line through the origin. Any two nonzero vectors in that line are necessarily scalar multiples of each other, hence collinear.

(c) Suppose that $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k} \in \mathbb{R}^n$ are linearly dependent, and let A be an $m \times n$ matrix. Prove that the vectors $A\vec{v_1}, A\vec{v_2}, \ldots, A\vec{v_k} \in \mathbb{R}^m$ are also linearly dependent.

Proof: Since $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$ are linearly dependent, there are scalars $c_1, \ldots, c_k$ not all equal to zero with

$$c_1\vec{v_1} + c_2\vec{v_2} + \ldots + c_k\vec{v_k} = \vec{0}.$$

Apply A to both sides of this equation:

$$A(c_1\vec{v_1} + c_2\vec{v_2} + \ldots + c_k\vec{v_k}) = A\vec{0}.$$

Using the properties of matrix-vector products on the left-hand side, and using that $A\vec{0} = \vec{0}$ on the right:

$$c_1A\vec{v_1} + c_2A\vec{v_2} + \ldots + c_kA\vec{v_k} = \vec{0}.$$

But since $c_1, \ldots, c_k$ are not all zero, this means that the vectors $A\vec{v_1}, A\vec{v_2}, \ldots, A\vec{v_k}$ are also linearly dependent.