SSEA Math 51 Track **Final Exam** **September 1st, 2011**

Please write your name and email address:

Name (please print clearly): ________________________________

Email address: ________________________________

- This is the final exam for the Math 51 track at SSEA. Answer as many problems as possible to the best of your ability; do not worry if you are not able to answer all of the problems. Partial credit is available. No calculators, notes, or other electronic devices are permitted.
- As with all math tests at Stanford, you are required to show your work in order to receive credit. In particular, you should not do computations in your head; instead, write them out on the test paper. You should also justify all conclusions that you make - do not be afraid to explain yourself by writing a sentence or two. The goal is to make your thought process as clear as possible.
- Please sign below to indicate your acceptance of the following statement:

  "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

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Problem	Total Points	Score
1	12	
2	12	
3	12	
4	10	
5	10	
6	15	
7	14	
8	15	
Total	100	

1 (12 points) Define the vectors $\vec{u}, \vec{v}$, and $\vec{w}$, and the matrices A, B, and C, as follows:

$$\vec{u} = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 3 & -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 2 \\ 3 & 1 \\ 2 & 0 \\ 1 & 4 \end{bmatrix}.$$

If the following quantities are defined, compute them; if not, explain why (points are 3/3/3/3):

(a) $A\vec{u}$.

(b) $B\vec{v}$.

(c) AC.

(d) BC.

2 (12 points) Complete the following definitions:

(a) A set of vectors $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ in $\mathbb{R}^n$ is *linearly independent* provided...

(b) The *span* of a set of vectors $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ in $\mathbb{R}^n$ is...

(c) The *null space* of an $m \times n$ matrix A is...

(d) The *rank* of an $m \times n$ matrix A is...

(e) A *basis* for a subspace V of $\mathbb{R}^n$ is a set of vectors $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ in V with the following properties...

(f) A *linear transformation* is a function T from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that the following two properties hold...

3 (12 points)

(a) Is each of these two subsets a subspace of $\mathbb{R}^2$? Explain why or why not.

i. The first quadrant; that is, the set of points (x, y) with $x \geq 0$ and $y \geq 0$.

ii. The line $y = 2x + 1$.

(b) Is each of these two functions a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$? Explain why or why not, and if it is a linear transformation, give the matrix:

i. $T(x_1, x_2) = (e^{x_1}, x_1 + x_2)$.

ii. $T(x_1, x_2) = (2x_1 + 3x_2, 7x_1 + 8x_2)$.

4 (10 points)

(a) Find a parametric representation for the line in $\mathbb{R}^4$ which passes through the points $(1, 2, -1, 2)$ and $(3, -2, 1, 0)$.

(b) Find a parametric representation for the plane in $\mathbb{R}^3$ which passes through the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

5 (10 points) Compute the reduced row echelon form of the following matrix:

$$A = \begin{bmatrix} 0 & 1 & -1 & 0 & 2 \\ 3 & 3 & 6 & 0 & 6 \\ 1 & 4 & -1 & -3 & -1 \\ -1 & 1 & -4 & 1 & 5 \end{bmatrix}.$$

6 (15 points) Consider the following matrix A and its reduced row echelon form rref(A); you do not need to check that the row reduction is correct:

$$A = \begin{bmatrix} 1 & 6 & -4 & 2 \\ 2 & -1 & 5 & 3 \\ -1 & -4 & 2 & 4 \\ 3 & 5 & 1 & 1 \\ -2 & -3 & -1 & 7 \end{bmatrix}; \; rref(A) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find a basis for the column space $C(A)$. What is the rank?

(b) Find a basis for the null space $N(A)$. What is the nullity?

(c) Let the vectors $\vec{u} \in \mathbb{R}^4$ and $\vec{b} \in \mathbb{R}^5$ be given by

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} -3 \\ 7 \\ 1 \\ 4 \\ -3 \end{bmatrix}.$$

Given that $A\vec{u} = \vec{b}$ (you do not need to check this), find the set of all solutions $\vec{x} \in \mathbb{R}^4$ to $A\vec{x} = \vec{b}$.

7 (14 points)

(a) (4 points) Find the matrix of the linear transformation T, where T is the linear transformation that leaves points on the y-axis fixed, but shifts a point (x, y) off the y-axis vertically by $-x$ units. (As we mentioned in class, this sort of transformation is called a *shear*.)

(b) (4 points) Find the matrices of the linear transformations R and S, where R is a rotation by $\pi/2$ radians in the *clockwise* direction and S is a rotation by $\pi/2$ radians in the counterclockwise direction. (Here S and R are said to be *inverses*.)

(c) (6 points) Compute the matrix of the transformation $R \circ T \circ S$. What kind of linear transformation is this?

8 (15 points)

(a) Suppose that A is a $m \times n$ matrix with rank n. Show that the equation $A\vec{x} = \vec{b}$ never has more than one solution, no matter which $\vec{b} \in \mathbb{R}^m$ we choose.

(b) Suppose that A is an $m \times n$ matrix with nullity equal to $n - 1$. Prove that any two nonzero vectors in $C(A)$ are collinear (that is, one is a scalar multiple of the other).

(c) Suppose that $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k} \in \mathbb{R}^n$ are linearly dependent, and let A be an $m \times n$ matrix. Prove that the vectors $A\vec{v_1}, A\vec{v_2}, \ldots, A\vec{v_k} \in \mathbb{R}^m$ are also linearly dependent.