SSEA Math 51 Track **Final Exam** **August 30, 2012**

Name: ____

- This is the final exam for the Math 51 track at SSEA. Answer as many problems as possible to the best of your ability; do not worry if you are not able to answer all of the problems. Partial credit is available. No calculators, notes, or other electronic devices are permitted.
- As with all math tests at Stanford, you are required to show your work in order to receive credit. In particular, you should not do computations in your head; instead, write them out on the test paper. You should also justify all conclusions that you make, and do not be afraid to explain yourself by writing a sentence or two. The goal is to make your thought process as clear as possible.
- Please sign below to indicate your acceptance of the following statement:

  "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ____

Problem	Total Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

1 Define vector v and matrices A and B as follows:

$$v = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 4 & 2 & 2 \\ -2 & -3 & 1 & 2 \\ 3 & 5 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 3 \\ 0 & -1 \\ 2 & 0 \end{bmatrix}.$$

(a) Find Av or explain why it is not defined.

Solution.

$$Av = \begin{bmatrix} 2 & 4 & 2 & 2 \\ -2 & -3 & 1 & 2 \\ 3 & 5 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -3 \end{bmatrix}.$$

(b) Find AB or explain why it is not defined.

Solution.

$$AB = \begin{bmatrix} 2 & 4 & 2 & 2 \\ -2 & -3 & 1 & 2 \\ 3 & 5 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 3 \\ 0 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 10 \\ 1 & -10 \\ 3 & 15 \end{bmatrix}.$$

(c) Find rref(A).

Solution.

$$A = \begin{bmatrix} 2 & 4 & 2 & 2 \\ -2 & -3 & 1 & 2 \\ 3 & 5 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ -2 & -3 & 1 & 2 \\ 3 & 5 & 0 & -1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & -1 & -3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & -7 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -5 & -7 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

2 (a) Find a parametric representation for the line in $\mathbb{R}^3$ which passes through the points $(1,2,0)$ and $(-1,3,3)$.

Solution.

Note

$$\begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}.$$

Hence one parametric representation of the line is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \middle| t \in \mathbb{R} \right\}.$$

(b) Find a parametric representation for the plane in $\mathbb{R}^4$ which passes through the points $(1,2,0,0)$, $(1,0,1,0)$, and $(0,1,0,1)$.

Solution.

Note

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence one parametric representation of the plane is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \middle| s, t \in \mathbb{R} \right\}.$$

3 Complete the following definitions:

(a) A set of vectors $\{v_1, v_2, \ldots, v_k\}$ in $\mathbb{R}^n$ is *linearly dependent* if ...

Solution.

...at least one of the vectors is a linear combination of the others. Or equivalently, if we can write

$$c_1v_1 + c_2v_2 + \ldots + c_kv_k = \vec{0}$$

with at least one of the coefficients c_i not equal to zero.

(b) Let A be an $m \times n$ matrix. The *column space* of a A is ...

Solution.

...the span of the columns of matrix A.

(c) Let A be an $m \times n$ matrix. The linear system $Ax = b$ is *homogeneous* if ...

Solution.

...$b = \vec{0}$.

(d) A set of vectors $\{v_1, v_2, \ldots, v_k\}$ in a linear subspace V is a *basis* for V if ...

Solution.

...$\{v_1, v_2, \ldots, v_k\}$ is linearly independent and $\text{span}(v_1, v_2, \ldots, v_k) = V$.

(e) The *dimension* of a nontrivial linear subspace V of $\mathbb{R}^n$ is ...

Solution.

...the number of elements in any basis for V.

4 (a) Suppose $\{u, v, w\}$ is a linearly independent set. Is $\{2u - v, u - v + w, u + v - 3w\}$ a linearly independent set? Show why or why not.

Solution.

To find out whether $\{2u - v, u - v + w, u + v - 3w\}$ is linearly independent, we consider the equation

$$c_1(2u - v) + c_2(u - v + w) + c_3(u + v - 3w) = \vec{0}.$$

Must all the coefficients be zero? Let's regroup the terms according to the vectors u, v, and w. We get

$$(2c_1 + c_2 + c_3)u + (-c_1 - c_2 + c_3)v + (c_2 - 3c_3)w = \vec{0}.$$

Since $\{u, v, w\}$ is linearly independent, the coefficients above must all be zero. That is, we have

$$\begin{aligned} 2c_1 + c_2 + c_3 &= 0 \\ -c_1 - c_2 + c_3 &= 0 \\ c_2 - 3c_3 &= 0. \end{aligned}$$

To solve this system of equations, we put the corresponding matrix in reduced row echelon form and get

$$\operatorname{rref}\left(\begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & -3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that c_3 is a free variable. Picking $c_3 = 1$, we get $c_1 = -2$ and $c_2 = 3$. Hence

$$-2(2u - v) + 3(u - v + w) + 1(u + v - 3w) = \vec{0}.$$

So $\{2u - v, u - v + w, u + v - 3w\}$ is a linearly dependent set.

(b) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Find conditions on the components of the vector b which are necessary and sufficient for b to be in the column space of matrix A.

Solution. We put the augmented matrix

$$[A \mid b] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 3 & 1 & b_2 \end{array}\right]$$

in reduced row echelon form. We get

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 3 & 1 & b_2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & -1 & -5 & b_2 - 2b_1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 1 & 5 & 2b_1 - b_2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -7 & b_1 - 2(2b_1 - b_2) \\ 0 & 1 & 5 & 2b_1 - b_2 \end{array}\right] = \operatorname{rref}([A \mid b]).$$

Since there are no rows that are entirely zero to the left of the augmentation, this means that there are no conditions on b_1 and b_2. Hence $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ is in the column space of A for any $b_1, b_2 \in \mathbb{R}$.

5 (a) Let V be the union of the x-axis and y-axis in $\mathbb{R}^2$. That is,

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; y = 0 \text{ or } x = 0 \right\}.$$

Is V a linear subspace? Explain why or why not.

Solution.

Note $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in V$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in V$ but

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin V.$$

Hence V is not a linear subspace.

(b) Let A be an $m \times n$ matrix. Prove that the null space $N(A)$ is a linear subspace of $\mathbb{R}^n$.

Solution.

We must show that $N(A)$ satisfies three properties.

i. Note $A\vec{0} = \vec{0}$, so $\vec{0} \in N(A)$.

ii. Suppose $x \in N(A)$ and $y \in N(A)$. So $Ax = \vec{0}$ and $Ay = \vec{0}$. Hence

$$A(x + y) = Ax + Ay = \vec{0} + \vec{0} = \vec{0}.$$

This shows that $x + y \in N(A)$.

iii. Suppose $x \in N(A)$ and $c \in \mathbb{R}$. So $Ax = \vec{0}$. Hence

$$A(cx) = cAx = c\vec{0} = \vec{0}.$$

This shows that $cx \in N(A)$.

Since the above three properties are satisfied, $N(A)$ is a linear subspace of $\mathbb{R}^n$.

6 (a) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the function given by $T(x_1, x_2) = (x_1, 2x_1 + x_2)$. Is T a linear transformation? Show why or why not. If T is linear, find the matrix A such that $T(x) = Ax$ for all $x \in \mathbb{R}^2$.

Solution.

We must show that T satisfies two properties.

i. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Note

$$\begin{aligned} T(x+y) &= T(x_1 + y_1, x_2 + y_2) \\ &= (x_1 + y_1, 2x_1 + 2y_1 + x_2 + y_2) \\ &= (x_1, 2x_1 + x_2) + (y_1, 2y_1 + y_2) \\ &= T(x) + T(y). \end{aligned}$$

ii. Let $x = (x_1, x_2)$ and $c \in \mathbb{R}$. Note

$$\begin{aligned} T(cx) &= T(cx_1, cx_2) \\ &= (cx_1, 2cx_1 + cx_2) \\ &= c(x_1, 2x_1 + x_2) \\ &= cT(x). \end{aligned}$$

Since the above two properties are satisfied, T is a linear transformation.

To find the matrix for T, note $T(e_1) = T(1,0) = (1,2)$ and $T(e_2) = T(0,1) = (0,1)$. Hence by Proposition 13.2, the matrix for T is

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

(b) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the function given by $T(x_1, x_2) = (0, x_1, x_1 x_2)$. Is T a linear transformation? Show why or why not. If T is linear, find the matrix A such that $T(x) = Ax$ for all $x \in \mathbb{R}^2$.

Solution.

Let $x = (1,1)$ and $c = 2$. Note

$$T(cx) = T(2,2) = (0,2,4)$$

but

$$cT(x) = 2T(1,1) = 2(0,1,1) = (0,2,2).$$

Since $T(cx) \neq cT(x)$, T is not a linear transformation.

7 Consider the matrix A and its reduced row echelon form rref(A):

$$A = \begin{bmatrix} 3 & 2 & 7 & 0 & 55 \\ 1 & 1 & 3 & -2 & 77 \\ 0 & 4 & 8 & -2 & 66 \end{bmatrix}; \quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & 17 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -29 \end{bmatrix}.$$

You do not need to check that the reduced row echelon form is correct.

(a) Find a basis for the column space $C(A)$. For every vector $b \in \mathbb{R}^3$, does there exist at least one solution x to the equation $Ax = b$?

Solution.

Note that rref(A) has pivots in the first, second, and fourth columns. By Proposition 11.2, the first, second, and fourth columns of A form a basis for $C(A)$. That is, one basis for $C(A)$ is

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix} \right\}.$$

Since rank$(A) = 3 =$ number of columns of A, for every vector $b \in \mathbb{R}^3$ there exists at least one solution x to the equation $Ax = b$.

(b) Find a basis for the null space $N(A)$. Is there ever a unique solution x to an equation of the form $Ax = b$?

Solution. The homogeneous system of equations corresponding to rref(A) and hence also to matrix A is

$$\begin{aligned} x_1 + x_3 + 17x_5 &= 0 \\ x_2 + 2x_3 + 2x_5 &= 0 \\ x_4 - 29x_5 &= 0. \end{aligned}$$

Solving for vector x in terms of the free variables, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -17 \\ -2 \\ 0 \\ 29 \\ 1 \end{bmatrix}.$$

Hence a basis for $N(A)$ is

$$\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -17 \\ -2 \\ 0 \\ 29 \\ 1 \end{bmatrix} \right\}.$$

Since nullity$(A) = 2 > 0$, there is never a unique solution x to an equation of the form $Ax = b$.

8 (a) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that reflects vectors through the xy-plane. Find the matrix B that satisfies $T(x) = Bx$ for all $x \in \mathbb{R}^3$.

Solution.

Note

$$T(e_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; T(e_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } T(e_3) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

Hence by Proposition 13.2 we have

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(b) Let $S\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the $90°$ rotation about the x-axis, where the direction is such that the positive y-axis is rotated toward the positive z-axis. Find the matrix A that satisfies $S(x) = Ax$ for all $x \in \mathbb{R}^3$.

Solution.

Note

$$S(e_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; S(e_2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ and } X(e_3) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

Hence by Proposition 13.2 we have

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

(c) Compute the matrix for the linear transformation $S \circ T$.

Solution.

By Proposition 15.3, the matrix for the linear transformation $S \circ T$ is

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

9 For the following true and false questions, you do not need to explain your answer at all. Just write "True" or "False".

(a) True or false: It is possible for a 3×4 matrix A to have $\text{rank}(A) = 4$ and $\text{nullity}(A) = 0$.

Solution.

False. Since the column space $C(A)$ is a subspace of $\mathbb{R}^3$, we must have $\text{rank}(A) = \dim(C(A)) \leq 3$.

(b) True or false: It is possible for a 3×4 matrix A to have $\text{rank}(A) = 0$ and $\text{nullity}(A) = 4$.

Solution.

True. Consider

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(c) True or false: For any matrix A we have $\text{rank}(A) = \text{rank}(\text{rref}(A))$.

Solution.

True. Let A be a $m \times n$ matrix. We know $N(A) = N(\text{rref}(A))$ by page 52. Hence $\text{nullity}(A) = \text{nullity}(\text{rref}(A))$. Using the Rank-Nullity Theorem, this implies that

$$\text{rank}(A) = n - \text{nullity}(A) = n - \text{nullity}(\text{rref}(A)) = \text{rank}(\text{rref}(A)).$$

(d) True or false: It is possible for $\text{span}\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$ to be a basis for the column space of a matrix.

Solution.

False. Note that $\text{span}\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$ contains the zero vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. However, a basis for a vector space must be a linearly independent set of vectors, and hence can never contain the zero vector.

(e) True or false: If $S = \{v_1, v_2, v_3\}$ is a linearly independent set of vectors, then every vector in S can be written as a linear combination of the other two vectors.

Note: I meant for this to instead say "linearly dependent," in which case the answer still would have been false. See Exercise 3.13

Solution.

False. If $S = \{v_1, v_2, v_3\}$ is a linearly independent set of vectors, then no vector in S can be written as a linear combination of the other two vectors.

10 (a) Write down a matrix A such that $N(A) = \text{span}\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)$. You do not need to show that your answer is correct.

Solution.

$$A = \begin{bmatrix} 1 & -3 \end{bmatrix}.$$

(b) Write down a matrix A such that $C(A) = \text{span}\left(\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$. You do not need to show that your answer is correct.

Solution.

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 3 & 0 \end{bmatrix}.$$

(c) Suppose that A is a 5×4 matrix, that $N(A) = \text{span}\left(\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$, and that

$$A \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}.$$

Write down the set of all solutions x to

$$Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}.$$

Solution.
Note that

$$x = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

is a particular solution to

$$Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}.$$

By Proposition 8.2, the set of all solutions is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} + r \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \,\middle|\, r, s, t \in \mathbb{R} \right\}.$$