SSEA Math 51 Track **Final Exam** **August 30, 2012**

Name: ____

- This is the final exam for the Math 51 track at SSEA. Answer as many problems as possible to the best of your ability; do not worry if you are not able to answer all of the problems. Partial credit is available. No calculators, notes, or other electronic devices are permitted.
- As with all math tests at Stanford, you are required to show your work in order to receive credit. In particular, you should not do computations in your head; instead, write them out on the test paper. You should also justify all conclusions that you make, and do not be afraid to explain yourself by writing a sentence or two. The goal is to make your thought process as clear as possible.
- Please sign below to indicate your acceptance of the following statement:

  "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ____

Problem	Total Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

1 Define vector v and matrices A and B as follows:

$$v = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \; A = \begin{bmatrix} 2 & 4 & 2 & 2 \\ -2 & -3 & 1 & 2 \\ 3 & 5 & 0 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 \\ 1 & 3 \\ 0 & -1 \\ 2 & 0 \end{bmatrix}.$$

(a) Find Av or explain why it is not defined.

(b) Find AB or explain why it is not defined.

(c) Find $\text{rref}(A)$.

2 (a) Find a parametric representation for the line in $\mathbb{R}^3$ which passes through the points $(1, 2, 0)$ and $(-1, 3, 3)$.

(b) Find a parametric representation for the plane in $\mathbb{R}^4$ which passes through the points $(1, 2, 0, 0)$, $(1, 0, 1, 0)$, and $(0, 1, 0, 1)$.

3 Complete the following definitions:

(a) A set of vectors $\{v_1, v_2, \ldots, v_k\}$ in $\mathbb{R}^n$ is *linearly dependent* if ...

(b) Let A be an $m \times n$ matrix. The *column space* of a A is ...

(c) Let A be an $m \times n$ matrix. The linear system $Ax = b$ is *homogeneous* if ...

(d) A set of vectors $\{v_1, v_2, \ldots, v_k\}$ in a linear subspace V is a *basis* for V if ...

(e) The *dimension* of a nontrivial linear subspace V of $\mathbb{R}^n$ is ...

4 (a) Suppose $\{u, v, w\}$ is a linearly independent set. Is $\{2u - v, u - v + w, u + v - 3w\}$ a linearly independent set? Show why or why not.

(b) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Find conditions on the components of the vector b which are necessary and sufficient for b to be in the column space of matrix A.

5 (a) Let V be the union of the x-axis and y-axis in $\mathbb{R}^2$. That is,

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; y = 0 \text{ or } x = 0 \right\}.$$

Is V a linear subspace? Explain why or why not.

(b) Let A be an $m \times n$ matrix. Prove that the null space $N(A)$ is a linear subspace of $\mathbb{R}^n$.

6 (a) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the function given by $T(x_1, x_2) = (x_1, 2x_1 + x_2)$. Is T a linear transformation? Show why or why not. If T is linear, find the matrix A such that $T(x) = Ax$ for all $x \in \mathbb{R}^2$.

(b) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the function given by $T(x_1, x_2) = (0, x_1, x_1x_2)$. Is T a linear transformation? Show why or why not. If T is linear, find the matrix A such that $T(x) = Ax$ for all $x \in \mathbb{R}^2$.

7 Consider the matrix A and its reduced row echelon form $\text{rref}(A)$:

$$A = \begin{bmatrix} 3 & 2 & 7 & 0 & 55 \\ 1 & 1 & 3 & -2 & 77 \\ 0 & 4 & 8 & -2 & 66 \end{bmatrix}; \quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & 17 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -29 \end{bmatrix}.$$

You do not need to check that the reduced row echelon form is correct.

(a) Find a basis for the column space $C(A)$. For every vector $b \in \mathbb{R}^3$, does there exist at least one solution x to the equation $Ax = b$?

(b) Find a basis for the null space $N(A)$. Is there ever a unique solution x to an equation of the form $Ax = b$?

8 (a) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that reflects vectors through the xy-plane. Find the matrix B that satisfies $T(x) = Bx$ for all $x \in \mathbb{R}^3$.

(b) Let $S\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the 90° rotation about the x-axis, where the direction is such that the positive y-axis is rotated toward the positive z-axis. Find the matrix A that satisfies $S(x) = Ax$ for all $x \in \mathbb{R}^3$.

(c) Compute the matrix for the linear transformation $S \circ T$.

9 For the following true and false questions, you do not need to explain your answer at all. Just write "True" or "False".

(a) True or false: It is possible for a 3×4 matrix A to have $\text{rank}(A) = 4$ and $\text{nullity}(A) = 0$.

(b) True or false: It is possible for a 3×4 matrix A to have $\text{rank}(A) = 0$ and $\text{nullity}(A) = 4$.

(c) True or false: For any matrix A we have $\text{rank}(A) = \text{rank}(\text{rref}(A))$.

(d) True or false: It is possible for $\text{span}\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$ to be a basis for the column space of a matrix.

(e) True or false: If $S = \{v_1, v_2, v_3\}$ is a linearly independent set of vectors, then every vector in S can be written as a linear combination of the other two vectors. *Note: I meant for this to instead say "linearly dependent," in which case the answer still would have been false. See Exercise 3.13*

10 (a) Write down a matrix A such that $N(A) = \text{span}\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)$. You do not need to show that your answer is correct.

(b) Write down a matrix A such that $C(A) = \text{span}\left(\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$. You do not need to show that your answer is correct.

(c) Suppose that A is a 5×4 matrix, that $N(A) = \text{span}\left(\begin{bmatrix}-3\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}4\\0\\0\\1\end{bmatrix}\right)$, and that

$$A\begin{bmatrix}2\\1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}.$$

Write down the set of all solutions x to

$$Ax = \begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}.$$