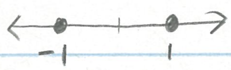
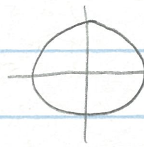
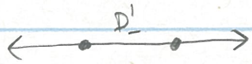
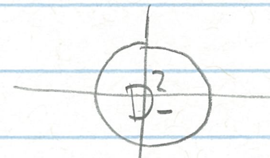
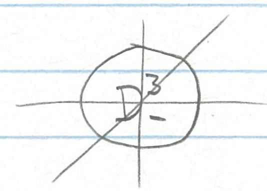
The n-dimensional sphere

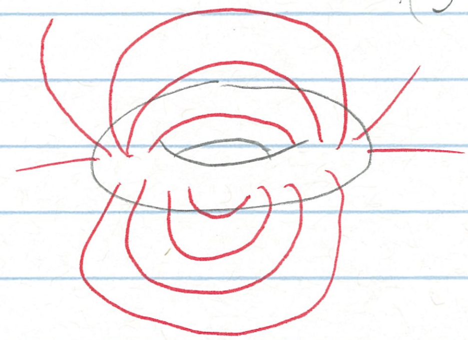
Def $S^n = \{ x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \} = D^n_+ \cup_{S^{n-1}} D^n_-$ is an n-dimensional manifold. $\left(D^n = \{ x \in \mathbb{R}^{n+1} \mid \|x\| \le 1 \} \right)$

n	0	1	2	3
S^n				?
S^n	∞ • $\mathbb{R}^0$ • D^0_-			

Can visualize S^3 as $\mathbb{R}^3$ plus a point at infinity!

Rmk The genus 1 Heegard splitting of S^3

$$S^3 = \partial(D^4) = \partial(D^2 \times D^2) = (\partial D^2 \times D^2) \cup (D^2 \times \partial D^2)$$
$$= (S^1 \times D^2) \cup_{S^1 \times S^1} (D^2 \times S^1)$$



$$\left(S^1 = \partial(D^2) = \partial(D^1 \times D^1) = (\partial D^1 \times D^1) \cup (D^1 \times \partial D^1) = (S^0 \times D^1) \cup (D^1 \times S^0) \right.$$

More generally, if $n+1 = p+q$, then
$$S^n = \partial(D^{n+1}) = \partial(D^{p+q}) = \partial(D^p \times D^q) = (\partial D^p \times D^q) \cup (D^p \times \partial D^q)$$
$$= (S^{p-1} \times D^q) \cup (D^p \times S^{q-1})$$

(See "Surgery theory" on wikipedia)

Ex $S^{2\ell+1} = (S^{2\ell-1} \times D^2) \cup (D^{2\ell} \times S^1)$