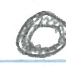
Introduction to Homotopy Equivalences & Homotopy Groups

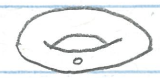
WARNING: These notes are intentionally non-rigorous (or incorrect!)

Big picture Let X, Y be topological spaces. They're homeomorphic $(X \cong Y)$ or homotopy equivalent $(X \simeq Y)$ if they have "the same shape"

"preserves dimension" doesn't "preserve dimension"

Ex  $\cong$  and  $\simeq$ 

Ex  $\not\cong$  but  $\simeq$ 

Ex  $\simeq$ 

Torus minus a point Wedge of two circles

- $X \cong Y \Rightarrow X \simeq Y$.

Big picture The homotopy group $\pi_i(X)$ $(i \geq 1)$ and homology group $H_i(X)$ $(i \geq 0)$ "measure the # of i-dimensional holes".

- $X \simeq Y \Longrightarrow \pi_i(X) = \pi_i(Y)$ and $H_i(X) = H_i(Y)$

isomorphisms of groups

- $\pi_1(X)$ is the fundamental group.
- $\pi_i(X)$ is easy to define & hard to compute
- $H_i(X)$ is hard to define & easy to compute (linear algebra).
- A map $f: X \to Y$ produces maps $\pi_i(X) \to \pi_i(Y)$ and $H_i(X) \to H_i(Y)$.

Jumping ahead

- $\pi_i(X)$ abelian for $i \geq 2$, $H_i(X)$ abelian $\forall i$
- $\pi_i(S^n) = \begin{cases} 0 & i < n \\ \mathbb{Z} & i = n \\ \text{often hard} & i > n \end{cases}$  $S^n = \{y \in \mathbb{R}^{n+1} \mid \|y\| = 1\}$

  $\pi_3(S^2) = \mathbb{Z}$ wild!
- $H_i(S^n) = \begin{cases} \mathbb{Z} & i = 0 \text{ or } n \\ 0 & \text{otherwise} \end{cases}$

- H_i (torus ) $= \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}\oplus\mathbb{Z}=\mathbb{Z}^2 & i=1 \\ \mathbb{Z} & i=2 \\ 0 & \text{otherwise} \end{cases}$ 
- $\pi_i(X\times Y) = \pi_i(X)\times\pi_i(Y)$
- π_i (torus $S^1\times S^1$) $= \begin{cases} \mathbb{Z}^2 & i=1 \\ 0 & \text{otherwise} \end{cases}$

All maps are continuous

Def $X\cong Y$ if $\exists$ continuous & bijective $f: X\to Y$
s.t. f^{-1} is continuous

Equivalently $X\cong Y$ if $\exists$ $X \underset{f^{-1}}{\overset{f}{\rightleftarrows}} Y$ s.t. $f^{-1}\circ f = id_X$, $f\circ f^{-1} = id_Y$.

Def $X\simeq Y$ if $\exists$ $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ s.t. $g\circ f\simeq id_X$, $f\circ g\simeq id_Y$

Need to define homotopy equivalences b/w maps

Ex ○ $\simeq$  since $\exists$ ○ $\underset{g}{\overset{f}{\rightleftarrows}}$  defined via

$1\le r\le 2$

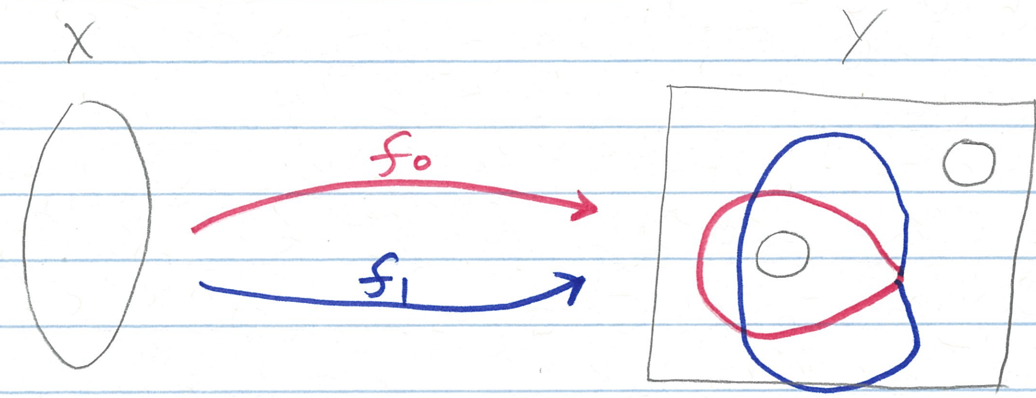
$f(e^{i\theta}) = e^{i\theta}$ $\qquad g(re^{i\theta}) = e^{i\theta}$

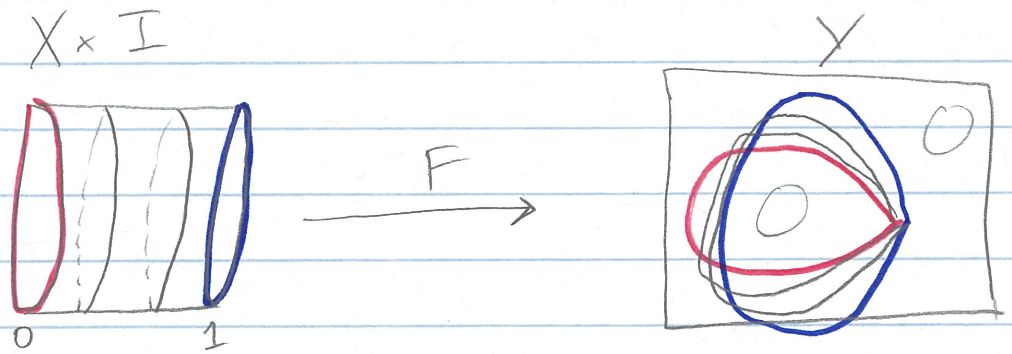
Note $g\circ f = id$ on circle.

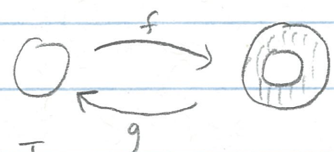
We'll see $f\circ g\simeq id$ on annulus.

Def Two maps $f_0, f_1 : X \to Y$ are homotopy equivalent (also denoted $f_0 \simeq f_1$) if $\exists$ continuous $F : X \times I \to Y$ with $F(\ ,0) = f_0$ and $F(\ ,1) = f_1$

Ex





Ex In  , show $f \circ g \simeq id$