Introduction to (simplicial) homology

Let X be a simplicial complex and G an abelian group.

Def Let C_i be the abelian group of all <u>i-chains</u>, i.e. the (formal) sums of i-simplices in X with coefficients in G.

Def Define $\partial_i : C_i \to C_{i-1}$ by

$$\partial_i([x_0, \ldots, x_i]) = \sum_{j=0}^{i} (-1)^j [x_0, \ldots, \hat{x}_j, \ldots, x_i]$$

and extending linearly.

$$\boxed{\cdots \to C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \to \cdots}$$

Def Let $Z_i = \ker \partial_i = \{ c \in C_i \mid \partial_i c = 0 \}$ be the set of all <u>i-cycles</u>.

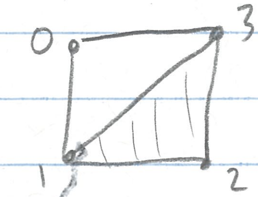
Def Let $B_i = \operatorname{im} \partial_{i+1} = \{ c \in C_i \mid c = \partial_{i+1} d \text{ for some } d \in C_{i+1} \}$ be the set of all <u>i-boundaries</u>.

Def $H_i(X; G) = Z_i / B_i$.

Properties

- $X \simeq Y \implies H_i(X; G) \cong H_i(Y; G)$
  homotopy equivalent — isomorphism of groups
- Homology is a functor: a continuous map of spaces $X \to Y$ induces a map of groups $H_i(X; G) \to H_i(Y; G)$.

Ex $X =$



$G = \mathbb{Z}/2\mathbb{Z} = \{0,1\}.$

$$C_1 = \{a[0,1] + b[0,3] + c[1,2] + d[1,3] + e[2,3] \mid a,b,c,d,e \in \mathbb{Z}/2\mathbb{Z}\}$$
$$= (\mathbb{Z}/2\mathbb{Z})^5$$

Group operation:

$$(a[0,1] + \dots + e[2,3]) + (a'[0,1] + \dots + e'[2,3]) = (a+a')[0,1] + \dots + (e+e')[2,3].$$

So $([0,1]+[1,3]) + ([1,3]+[0,3]) = [0,1] + 2[1,3] + [0,3]$
$= [0,1] + [0,3].$

$\partial_1 : C_1 \to C_0$ via $\partial_1([x_0,x_1]) = [x_1] + [x_0]$

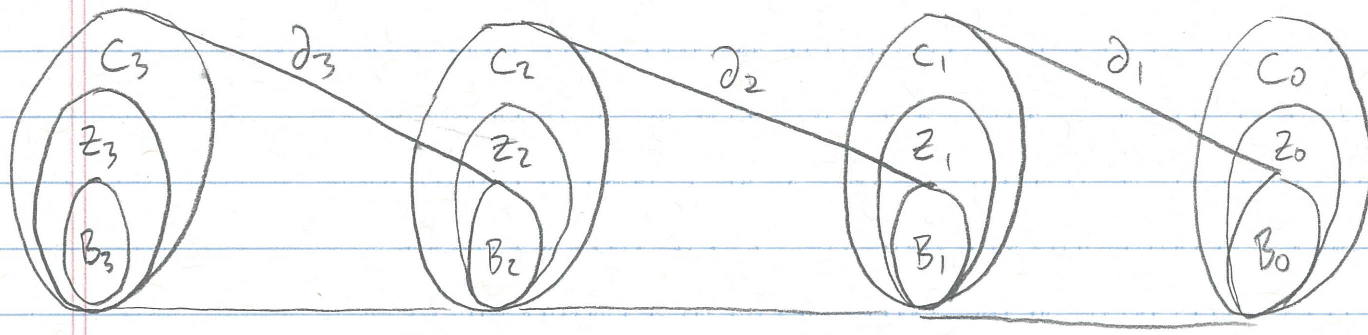
$\partial_1([0,1]+[1,3]) = \partial_1([0,1]) + \partial_1([1,3]) = ([1]+[0]) + ([3]+[1]) = [0]+[3].$

$\partial_2 : C_2 \to C_1$ via $\partial_2([1,2,3]) = [2,3] + [1,3] + [1,2]$

$$Z_1 = \ker \partial_1 = \{\phi, [0,1]+[1,3]+[3,0], [1,2]+[2,3]+[3,1], [0,1]+[1,2]+[2,3]+[3,0]\}$$
$$= \{a([0,1]+[1,3]+[3,0]) + b([1,2]+[2,3]+[3,1]) \mid a,b \in \mathbb{Z}/2\mathbb{Z}\}.$$
$$= (\mathbb{Z}/2\mathbb{Z})^2$$

$$B_1 = \operatorname{im} \partial_2 = \{\phi, \partial([1,2,3])\} = \{\phi, [2,3]+[1,3]+[1,2]\}$$
$$= \mathbb{Z}/2\mathbb{Z}$$

$$H_1(X; \mathbb{Z}/2\mathbb{Z}) = Z_1/B_1 = \mathbb{Z}/2\mathbb{Z}$$



Previous example

$C_3 = 0$ $\qquad \partial_3 = 0$ $\qquad C_2 = \mathbb{Z}/2\mathbb{Z}$ $\qquad C_1 = (\mathbb{Z}/2\mathbb{Z})^5$ $\qquad C_0 = (\mathbb{Z}/2\mathbb{Z})^4$

$Z_3 = 0$ $\qquad Z_2 = 0$ $\qquad Z_1 = (\mathbb{Z}/2\mathbb{Z})^2$ $\qquad Z_0 = (\mathbb{Z}/2\mathbb{Z})^4$

$B_3 = 0$ $\qquad B_2 = 0$ $\qquad B_1 = \mathbb{Z}/2\mathbb{Z}$ $\qquad B_0 = (\mathbb{Z}/2\mathbb{Z})^3$

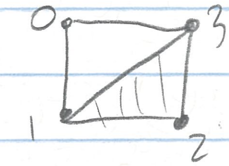
∂_2 has rank 1 nullity 0

∂_1 has rank 3 nullity 2

$$H_2(X; \mathbb{Z}/2\mathbb{Z}) = Z_2/B_2 = 0$$

$$H_1(X; \mathbb{Z}/2\mathbb{Z}) = Z_1/B_1 = \mathbb{Z}/2\mathbb{Z}$$

$$H_0(X; \mathbb{Z}/2\mathbb{Z}) = Z_0/B_0 = \mathbb{Z}/2\mathbb{Z}$$

Ex $X =$ 

$G = \mathbb{Z}$

$$C_1 = \{a[0,1] + b[0,3] + c[1,2] + d[1,3] + e[2,3] \mid a,b,c,d,e \in \mathbb{Z}\} = \mathbb{Z}^5$$

So $([0,1]+[1,3]) + ([1,3]+[0,3]) = [0,1] + 2[1,3] + [0,3]$

$\partial_1 : C_1 \to C_0$ via $\partial_1([x_0, x_1]) = [x_1] - [x_0]$

$\partial_1([0,1]+[1,3]) = \partial_1([0,1]) + \partial_1([1,3]) = [1]-[0]+([3]-[1]) = [3]-[0]$

$\partial_2 : C_2 \to C_1$ via $\partial_2([1,2,3]) = [2,3] - [1,3] + [1,2]$

$= [2,3] + [3,1] + [1,2]$

$$Z_1 = \ker \partial_1 = \{a([0,1]+[1,3]+[3,0]) + b([1,2]+[2,3]+[3,1]) \mid a,b \in \mathbb{Z}\}$$
$$= \mathbb{Z}^2$$

$$B_1 = \operatorname{im} \partial_2 = \{a([2,3]+[3,1]+[1,2]) \mid a \in \mathbb{Z}\} = \mathbb{Z}$$

$$H_1(X;\mathbb{Z}) = Z_1/B_1 = \mathbb{Z}$$