Math 51, Winter 2013
Henry Adams, February 26

These are (most likely) the problems we will do together during our review session for midterm 2.

(1) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation by $30°$ counterclockwise. Let $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a reflection across the line $y = x$.
   (a) Find matrix B satisfying $T(x) = Bx$.
   (b) Find matrix A satisfying $S(x) = Ax$.
   (c) Find the matrix corresponding to the linear transformation $S \circ T$.

(2) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation satisfying

$$T\left(\begin{bmatrix}1/2\\1/2\end{bmatrix}\right) = \begin{bmatrix}-3\\1\\0\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}1/2\\-1/2\end{bmatrix}\right) = \begin{bmatrix}0\\1\\2\end{bmatrix}.$$

   Find matrix A such that $T(x) = Ax$ for all $x \in \mathbb{R}^2$.

(3) Let A be an invertible matrix. Suppose v is an eigenvector of A with eigenvalue 3. Show v is also an eigenvector of $(A^3 + 2I + 4A^{-1})$ and find the corresponding eigenvalue.

(4) Let $g(t) = \left(e^t, \cos t^2, \ln(t+1)\right)$. At the point $(1,1,0)$ on the curve, find the velocity vector, the acceleration vector, the speed, and the tangent line to the curve.

(5) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear projection onto the line $y = 3x$.
   (a) Find an eigenbasis $\mathcal{B}$ for T.
   (b) Find the matrix B for T with respect to your eigenbasis $\mathcal{B}$.
   (c) Find the matrix A for T with respect to the standard basis.

(6) Let $A = \begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}$.
   (a) Find the eigenvalues for A and the corresponding eigenspaces.
   (b) Find an eigenbasis for A.
   (c) Find A^{20}.
   (d) Is matrix A invertible?

(7) Match each function with its contour map (not drawn). Give a brief mathematical justification for each matching.
   (a) $a(x,y) = e^x \cos y$.
   (b) $b(x,y) = y^4/x$.
   (c) $c(x,y) = \frac{x-y}{1+x^2+y^2}$.
   (d) $d(x,y) = |x+y|$.

(8) Let $V = \text{span}(v_1, v_2)$ be a subspace in $\mathbb{R}^3$, where $v_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$ and $v_2 = \begin{bmatrix}0\\1\\-1\end{bmatrix}$. Note $\mathcal{B} = \{v_1, v_2\}$ is a basis for V.
   (a) If $w \in V$ has $[w]_{\mathcal{B}} = \begin{bmatrix}2\\3\end{bmatrix}$, find w.
   (b) If $u = \begin{bmatrix}-1\\0\\-5\end{bmatrix} \in V$, find $[u]_{\mathcal{B}}$.

(9) Find the definiteness of the quadratic form $Q(x,y,z) = x^2 + 2y^2 + 6z^2 + 6xz$.

(10) True or False? Explain your answer
   (a) If $\det(\lambda I - A) = (\lambda - 4)^2(\lambda - 5)$ then matrix A is diagonalizable.
   (b) If $A^T = A$ then there exists a matrix C and a diagonal matrix B such that $A = CBC^{-1}$.
   (c) Quadratic form $Q(x,y) = x^2 + y^2 + 4xy$ is positive definite.
   (d) There exist similar matrices A and B with $\det(A) = -3$ and $\det(B) = 5$.
   (e) Matrix $A = \begin{bmatrix}3&0&2\\-1&0&1\\0&2&0\end{bmatrix}$ is invertible.