Math 51, Winter 2013
Henry Adams, January 30

These are the problems we will do together during our review session for midterm 1.

(1) (a) For any vectors $x$ and $y$ in $\mathbb{R}^n$, show that $x \cdot y = y \cdot x$ .
(b) Suppose $v \in \mathbb{R}^n$ is a unit vector. Prove that for any vector $w \in \mathbb{R}^n$, the vector $w - (w \cdot v)v$ is orthogonal to $v$.

(2) Let $A$ be an $m \times n$ matrix, and suppose that $\{v_1, v_2, \ldots, v_k\}$ is a linearly dependent set of vectors in $\mathbb{R}^n$. Prove that the set of vectors $\{Av_1, Av_2, \ldots, Av_k\}$ in $\mathbb{R}^m$ is also linearly dependent.

(3) Let $A = \begin{bmatrix} 3 & 0 & -3 & 12 & 0 \\ 1 & 2 & 3 & 10 & 0 \\ 2 & 1 & 0 & 11 & 1 \\ 0 & 1 & 2 & 3 & 1 \end{bmatrix}$.
(a) Find $\text{rref}(A)$.
(b) Find $\text{nullity}(A)$ and $\text{rank}(A)$.
(c) Express $N(A)$ in parametric form, express $N(A)$ as a span of linearly indpendent vectors, and find a basis for $N(A)$.
(d) Find a basis for $C(A)$.
(e) Find all solutions $x$ to $Ax = Ac$, where $c = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$.
(f) Is there a vector $b \in \mathbb{R}^4$ such that $Ax = b$ has no solutions $x$?
(g) Is there a vector $b \in \mathbb{R}^4$ such that $Ax = b$ has exactly one solution $x$?

(4) Let $A$ be an $m \times n$ matrix. Prove that the null space $N(A)$ is a linear subspace of $\mathbb{R}^n$.

(5) Find a parametric equation for the plane in $\mathbb{R}^4$ that passes through the points $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 5 \\ 6 \\ 7 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ -2 \\ 3 \end{bmatrix}$.

(6) True or false? Explain your answer.
(a) It is possible for a $2 \times 3$ matrix $A$ to have $\text{rank}(A) = 3$ and $\text{nullity}(A) = 0$.
(b) It is possible for a $3 \times 4$ matrix $A$ to have $\text{rank}(A) = 0$ and $\text{nullity}(A) = 4$.
(c) For every matrix $A$ we have $N(A) = N(\text{rref}(A))$.
(d) For every matrix $A$ we have $C(A) = C(\text{rref}(A))$.
(e) For every matrix $A$ we have $\text{rank}(A) = \text{rank}(\text{rref}(A))$.
(f) If $S = \{v_1, v_2, v_3\}$ is a linearly dependent set of vectors, then every vector in $S$ can be written as a linear combination of the other two vectors.
(g) If $V$ is subspace with $\dim(V) = 3$ and if $\{v_1, v_2, v_3\}$ is a linearly independent set of vectors in $V$, then $\{v_1, v_2, v_3\}$ is a basis for $V$.

(7) Let $\{u, v, w\}$ be a basis for a subspace $V$ of $R^n$. Is $\{u - v, v - w, u - w\}$ a basis for $V$?

(8) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Find conditions on the components of vector $b$ which are necessary and sufficient for $b$ to be in the column space of matrix $A$.

(9) Find all vectors which are orthogonal to both $\begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$.

(10) Suppose $v_1$, $v_2$, and $v_3$ are nonzero vectors that are pairwise orthogonal (meaning each vector is orthogonal to each of the other two). Prove that the set $\{v_1, v_2, v_3\}$ is linearly independent.

[This is a hard problem, in my opinion harder than the proofs you're likely to see.]