Math 51, Winter 2013
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Lagrange multipliers example

Problem: Find the global extrema of $f(x,y) = 2x + 3y$ on the set $S = \{(x,y) \mid x^2 + 2y^2 = 34\}$.

Answer: Define $g(x,y) = x^2 + 2y^2$, and note that $S = g^{-1}(34)$. By Theorem 26, any local extrema of f on S satisfies either $\nabla g(x,y) = \vec{0}$ or $\nabla f(x,y) = \lambda \nabla g(x,y)$.

Which candidate extrema in S satisfy $\nabla g(x,y) = \vec{0}$? Since $\nabla g(x,y) = \begin{bmatrix} 2x \\ 4y \end{bmatrix}$, we need $2x = 0$ and $4y = 0$, that is, we need $(x,y) = (0,0)$. But note that $(0,0)$ is not in $S = g^{-1}(34)$, since $g(0,0) = 0 \neq 34$. Hence no candidate extrema in S satisfy $\nabla g(x,y) = \vec{0}$.

Which candidate extrema in S satisfy $\nabla f(x,y) = \lambda \nabla g(x,y)$? Since $\nabla f(x,y) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\nabla g(x,y) = \begin{bmatrix} 2x \\ 4y \end{bmatrix}$, we get the following system of three equations:

$$\begin{aligned} 2 &= \lambda 2x \\ 3 &= \lambda 4y \\ x^2 + 2y^2 &= 34 \end{aligned}$$

We include the third equation because we are only searching for the extrema of f on set S.

Note that $x \neq 0$, for otherwise the first equation would be false. Hence we can divide by $2x$ in the first equation to get $\frac{1}{x} = \lambda$. Similarly, $y \neq 0$, for otherwise the second equation would be false. Hence we can divide by $4y$ in the second equation to get $\frac{3}{4y} = \lambda$.

We have $\frac{1}{x} = \lambda = \frac{3}{4y}$, so $x = \frac{4}{3}y$. Plugging this into the third equation, we get

$$\left(\frac{4}{3}y\right)^2 + 2y^2 = 34 \quad \Longrightarrow \quad \frac{34}{9}y^2 = 34 \quad \Longrightarrow \quad y^2 = 9 \quad \Longrightarrow \quad y = 3 \text{ or } -3.$$

From the equation $x = \frac{4}{3}y$, we see that if $y = 3$ then $x = 4$, and if $y = -3$ then $x = -4$. So our two candidate extrema are $(x,y) = (4,3)$ and $(x,y) = (-4,-3)$.

Since set S is closed and bounded and function f is continuous, Theorem 25 tells us that f attains its global extrema on S. The only candidate extrema are $(4,3)$ and $(-4,-3)$, so the global extrema must be among these candidates. We compare the values $f(4,3) = 17$ and $f(-4,-3) = -17$ to see that the global maximum of f on S occurs at $(4,3)$ and has value 17 and that the global minimum of f on S occurs at $(-4,-3)$ and has value -17. Boom!