Math 51, Winter 2013
Henry Adams, March 16

These are (most likely) the problems we will do together during our review session for the final.

(1) Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let $V = \{\vec{x} \in \mathbb{R}^n \mid T(\vec{x}) = 3\vec{x}\}$. Prove that V is a linear subspace of $\mathbb{R}^n$.

(2) (a) Let $g(x) = (\cos x, e^{-x})$. Suppose $f\colon \mathbb{R}^3 \to \mathbb{R}$ has $Df(2,1,1) = \begin{bmatrix} 3 & 0 & 1 \end{bmatrix}$ and $f(2,1,1) = 0$. Find $D(g \circ f)(2,1,1)$.

(b) Let $g(x) = (\cos x, e^{-x})$. Suppose $h\colon \mathbb{R}^2 \to \mathbb{R}^3$ has

$$Dh(1,1) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & 3 \end{bmatrix}.$$

Find $D(h \circ g)(0)$.

(3) Suppose $V(r,h) = \pi r^2 h$ where r and h are functions of t. Suppose at time $t = 7$ we have $r(7) = 2$, $h(7) = 6$, $\frac{\partial r}{\partial t}(7) = -1$, and $\frac{\partial h}{\partial t}(7) = -3$. Find $\frac{\partial V}{\partial t}(7)$.

(4) Let $f(x,y) = 2x^2 + 4xy - y^2 - 3x$.

(a) Find the critical points of f and classify each as a local minima, local maxima, or saddle point.

(b) Show that f has no global extrema on $\mathbb{R}^2$.

(c) Find the linearization of f at $(1/4, 1/2)$.

(d) Find the second-order Taylor approximation of f at $(1/4, 1/2)$.

(5) Let $f(x,y) = x^2 + xy - 2y$ and $D = \{(x,y)\} \mid -5 \le x \le 5 \text{ and } -5 \le y \le 5\}$.

(a) Explain why f has global extrema on D.

(b) Find the global extrema of f on D.

(6) Find the shortest distance from the curve $xy^2 = 2$ to the origin. You may assume that there is a point on this curve where the distance is minimized.

(7) (a) Consider the surface $S = \{(x,y,z) \mid xyz = 8\}$. Find the tangent plane to S at $(4,-1,-2)$.

(b) Let $f(x,y) = x^3 y + y^2$. Find the tangent plane to the graph of f at $(2,-1,-7)$.

(8) Let $\mathcal{B} = \{v_1, v_2, v_3\}$ be a basis for $\mathbb{R}^3$. Suppose linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $T(v_1) = v_2 + v_3$, $T(v_2) = v_3$, and $T(v_3) = \vec{0}$.

(a) Find matrix B for T with respect to basis $\mathcal{B}$.

(b) Let A be the matrix for T with respect to the standard basis. Show that

$$A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(9) Let $h(x,y) = \sin(x - 2y) + 2\cos(x + 3y) + 4$ be the temperature at point (x,y) in the plane. You are standing at the point $(\pi/6, \pi/6)$.

(a) In which direction should you start moving to decrease your temperature as fast as possible?

(b) If you start moving in the direction

$$\begin{bmatrix} -1 \\ -1 \end{bmatrix},$$

does your temperature initially increase or decrease?

(10) True or false? Provide justification.

(a) If A is a 2×3 matrix, then no solution x to $Ax = b$ is unique.

(b) If A is a 3×2 matrix, then no solution x to $Ax = b$ is unique.

(c) If A is a 2×3 matrix, then there is always a solution x to $Ax = b$.

(d) If A is an invertible matrix, then there is always a solution x to $Ax = b$.

(e) If $f\colon \mathbb{R}^n \to \mathbb{R}$ is differentiable and $a \in \mathbb{R}^n$, then $D_{\nabla f(a)} f(a) = \|\nabla f(a)\|^2$.