Math 51, Winter 2013
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Example: Is this matrix diagonalizable?

Problem: Let

$$A = \begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}.$$

Is matrix A diagonalizable?

Answer: By Proposition 23.2, matrix A is diagonalizable if and only if there is a basis of $\mathbb{R}^3$ consisting of eigenvectors of A. So let's find the eigenvalues and eigenspaces for matrix A.

By Proposition 23.1, λ is an eigenvalue of A precisely when $\det(\lambda I - A) = 0$. Note

$$\lambda I - A = \begin{bmatrix} \lambda - 6 & -3 & 8 \\ 0 & \lambda + 2 & 0 \\ -1 & 0 & \lambda + 3 \end{bmatrix}.$$

To find $\det(\lambda I - A)$ let's do cofactor expansion along the second row because it has many zeros[1]. We get

$$\begin{aligned}
\det(\lambda I - A) &= -0 \begin{vmatrix} -3 & 8 \\ 0 & \lambda + 3 \end{vmatrix} + (\lambda + 2) \begin{vmatrix} \lambda - 6 & 8 \\ -1 & \lambda + 3 \end{vmatrix} - 0 \begin{vmatrix} \lambda - 6 & -3 \\ -1 & 0 \end{vmatrix} \\
&= (\lambda + 2) \begin{vmatrix} \lambda - 6 & 8 \\ -1 & \lambda + 3 \end{vmatrix} \\
&= (\lambda + 2)\Big((\lambda - 6)(\lambda + 3) - 8(-1)\Big) \\
&= (\lambda + 2)\big(\lambda^2 - 3\lambda - 10\big) \\
&= (\lambda + 2)(\lambda + 2)(\lambda - 5) \\
&= (\lambda + 2)^2(\lambda - 5).
\end{aligned}$$

Hence our eigenvalues are $\lambda = -2$ and $\lambda = 5$. Note that $\lambda = -2$ is a repeated root with multiplicity two.

We find the corresponding eigenspaces. We get that

$$E_{-2} = N\left(\begin{bmatrix} -8 & -3 & 8 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}\right) = \ldots = N\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \ldots = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$$

and

$$E_5 = N\left(\begin{bmatrix} -1 & -3 & 8 \\ 0 & 7 & 0 \\ -1 & 0 & 8 \end{bmatrix}\right) = \ldots = N\left(\begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \ldots = \operatorname{span}\left(\begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix}\right).$$

The ellipses show where I have omitted work that you should know how to do, namely putting a matrix in reduced row echelon form and writing a null space as a span.

We have found only two linearly independent eigenvectors for A, namely the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix}.$$

But any basis for $\mathbb{R}^3$ consists of three vectors. Therefore there is no eigenbasis for A, and so by Proposition 23.2 matrix A is not diagonalizable.

Remark: The reason why matrix A is not diagonalizable is because the dimension of E_{-2} (which is 1) is smaller than the multiplicity of eigenvalue $\lambda = -2$ (which is 2).

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[1]In section we did cofactor expansion along the first column, which also works, but makes the resulting cubic polynomial harder to factor.