Math 51, Winter 2013
Henry Adams, January 22

Example: Column Space Conditions

Problem: Let

$$A = \begin{bmatrix} 1 & 3 & -1 & 9 \\ 1 & 1 & 3 & 1 \\ 2 & 7 & -4 & 22 \\ 3 & 8 & -1 & 23 \end{bmatrix} \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

Find conditions on the components of vector b which are necessary and sufficient for b to be in $C(A)$.

Answer: Recall that b is in $C(A)$ if the system of equations $Ax = b$ has at least one solution x. To analyze this system of equations, let's take the corresponding augmented matrix and put it in reduced row echelon form.

Here's the augmented matrix.

$$\left[\begin{array}{cccc|c} 1 & 3 & -1 & 9 & b_1 \\ 1 & 1 & 3 & 1 & b_2 \\ 2 & 7 & -4 & 22 & b_3 \\ 3 & 8 & -1 & 23 & b_4 \end{array}\right]$$

We subtract the first row from the second, subtract two times the first row from the third, and subtract three times the first row from the fourth.

$$\left[\begin{array}{cccc|c} 1 & 3 & -1 & 9 & b_1 \\ 0 & -2 & 4 & -8 & b_2 - b_1 \\ 0 & 1 & -2 & 4 & b_3 - 2b_1 \\ 0 & -1 & 2 & -4 & b_4 - 3b_1 \end{array}\right]$$

We swap the second and third rows.

$$\left[\begin{array}{cccc|c} 1 & 3 & -1 & 9 & b_1 \\ 0 & 1 & -2 & 4 & b_3 - 2b_1 \\ 0 & -2 & 4 & -8 & b_2 - b_1 \\ 0 & -1 & 2 & -4 & b_4 - 3b_1 \end{array}\right]$$

We subtract three times the second row from the first, add two times the second row to the third, and add the second row to the fourth.

$$\left[\begin{array}{cccc|c} 1 & 0 & 5 & -3 & b_1 - 3(b_3 - 2b_1) \\ 0 & 1 & -2 & 4 & b_3 - 2b_1 \\ 0 & 0 & 0 & 0 & b_2 - b_1 + 2(b_3 - 2b_1) \\ 0 & 0 & 0 & 0 & b_4 - 3b_1 + (b_3 - 2b_1) \end{array}\right]$$

We combine terms in the augmented column.

$$\left[\begin{array}{cccc|c} 1 & 0 & 5 & -3 & -b_1 - 3b_3 \\ 0 & 1 & -2 & 4 & b_3 - 2b_1 \\ 0 & 0 & 0 & 0 & -5b_1 + b_2 + 2b_3 \\ 0 & 0 & 0 & 0 & -5b_1 + b_3 + b_4 \end{array}\right]$$

Now we're ready to answer the question. Recall that b is in $C(A)$ if the system of equations $Ax = b$ has at least one solution x. And this system has at least one solution so long as as we have no inconsistent equations of the form

$$0 = 1 \quad \text{or} \quad 0 = \text{any constant other than } 0.$$

Hence this system of equations has at least one solution so long as

$$0 = -5b_1 + b_2 + 2b_3 \quad \text{and} \quad 0 = -5b_1 + b_3 + b_4.$$

In summary, b is in $C(A)$ if and only if

$$0 = -5b_1 + b_2 + 2b_3 \quad \text{and} \quad 0 = -5b_1 + b_3 + b_4.$$